CO-UNIVERSAL ALGEBRAS ASSOCIATED TO PRODUCT SYSTEMS, AND GAUGE-INVARIANT UNIQUENESS THEOREMS

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Abstract. Let X be a product system over a quasi-lattice ordered group. Under mild hypotheses, we associate to X a C^*-algebra which is co-universal for injective Nica covariant Toeplitz representations of X which preserve the gauge coaction. Under appropriate amenability criteria, this co-universal C^*-algebra coincides with the Cuntz-Nica-Pimsner algebra introduced by Sims and Yeend. We prove two key uniqueness theorems, and indicate how to use our theorems to realise a number of reduced crossed products as instances of our co-universal algebras. In each case, it is an easy corollary that the Cuntz-Nica-Pimsner algebra is isomorphic to the corresponding full crossed product.

1. Introduction

In the late 1970s and early 1980s, Cuntz and Krieger introduced a class of simple C^*-algebras generated by partial isometries, now known as the Cuntz-Krieger algebras [7, 8]. In 1997, Pimsner developed a far-reaching generalisation of Cuntz and Krieger's construction by associating to each right-Hilbert A–A bimodule X (also known as a C^*-correspondence over A) two C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$ (see [27]). The algebras $\mathcal{O}_X$ are direct generalisations of the Cuntz-Krieger algebras, and are now known as Cuntz-Pimsner algebras while the algebras $\mathcal{T}_X$ are generalisations of their Toeplitz extensions.

Pimsner showed that his construction also generalises crossed products by $\mathbb{Z}$. If α is an automorphism of a C^*-algebra A, there is a standard way to view A as a right-Hilbert A–A bimodule, denoted $X = {}_\alpha A$, and the Cuntz-Pimsner algebra $\mathcal{O}_X$ is isomorphic to the crossed product $A \times_\alpha \mathbb{Z}$. In general, it makes sense to think of right-Hilbert A–A bimodules as generalised endomorphisms of A so that $\mathcal{T}_X$ is like a crossed product of A by $\mathbb{N}$ and $\mathcal{O}_X$ is then like a crossed product by $\mathbb{Z}$.

In an impressive array of papers [21, 22, 23] Katsura, drawing on insight from graph algebras contained in [19], not only expanded Pimsner's theory of C^*-algebras associated with Hilbert bimodules beyond the case of isometric left actions, but also unified the theory of graph algebras and homeomorphism C^*-algebras under the term topological graphs, and proved (among other things) uniqueness theorems for his algebras. For $\mathcal{O}_X$ his result says that a Cuntz-Pimsner representation of the bimodule X generates an isomorphic copy of $\mathcal{O}_X$ precisely when the representation is injective and admits a gauge action. This type of result, due in genesis to an Huef and Raeburn [20], is now

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commonly known as "the gauge-invariant uniqueness theorem," and has proven to be a powerful tool for studying analogues of Cuntz-Krieger algebras.

In another direction that also generalises Pimsner's work, Fowler introduced and studied C^*-algebras associated to product systems of Hilbert bimodules [18]. Product systems over $(0, \infty)$ of Hilbert spaces were introduced and studied by Arveson [2], and the concept was later generalised to other semigroups by Dinh [9] and Fowler [17]. Just as a right-Hilbert A–A bimodule can be thought of as a generalised endomorphism, a product system over a semigroup P of right-Hilbert A–A bimodules can be regarded as an action of P on A by generalised endomorphisms. To study C^*-algebras associated to such objects, Fowler followed the lead of Nica [26] who had developed Toeplitz-type algebras for nonabelian semigroups P which embed in a group G in such a way as to induce what he called a quasi-lattice order. Based on Nica's formulation, Fowler associated C^*-algebras $\mathcal{T}_{\text{cov}}(X)$ to what he called compactly aligned product systems X over quasi-lattice ordered groups (G, P) and established that $\mathcal{T}_{\text{cov}}(X)$ had much of the structure of a twisted crossed product of the coefficient algebra A by the semigroup P, with the "twist" coming from X.

Fowler also associated to each product system a generalised Cuntz-Pimsner algebra $\mathcal{O}_X$. However, $\mathcal{O}_X$ need not in general be a quotient of $\mathcal{T}_{\text{cov}}(X)$. Moreover, the canonical homomorphism from A to $\mathcal{O}_X$ is, in general, not injective, so there is little hope of a gauge-invariant uniqueness theorem. Recently, however, Sims and Yeend [35] introduced what they call the Cuntz-Nica-Pimsner algebra of a product system X over a quasi-lattice ordered group (G, P). This algebra is a quotient of $\mathcal{T}_{\text{cov}}(X)$, and Sims and Yeend established that under relatively mild hypotheses the canonical representation of the product system X on the Cuntz-Nica-Pimsner algebra $\mathcal{N}\mathcal{O}_X$ is isometric. However, they were unable to establish a gauge-invariant uniqueness theorem for $\mathcal{N}\mathcal{O}_X$.

The initial purpose of the research presented in this article was to understand and describe the fixed-point algebra, called the *core*, for the canonical coaction of G on $\mathcal{N}\mathcal{O}_X$, and use this analysis to establish the missing gauge-invariant uniqueness theorem. Since $\mathcal{N}\mathcal{O}_X$ is defined for pairs (G, P) in which G need not be abelian, Katsura's gauge action of $\mathbb{T}$, equivalently seen as a coaction of $\mathbb{Z}$, must be replaced by a coaction of G.

We analyse the core in Section 3 and we subsequently prove a gauge-invariant uniqueness theorem in Corollary 4.11. This result is quite far-reaching in itself: in particular, it enables us to recover isomorphisms of various full and reduced crossed products — ordinary or partial — in the presence of amenability (see Section 5). More importantly, for the class of topological higher-rank graphs introduced by Yeend [37] in his generalisation of Katsura's topological graphs from [21], the result is new, and its proof follows a rather different path than earlier proofs in other contexts involving product systems over $\mathbb{N}^k$.

However, we do not proceed from the analysis of the core to the gauge-invariant uniqueness theorem in the usual fashion, and the main thrust of our results in the later sections of the paper deals with what happens when $\mathcal{N}\mathcal{O}_X$ does not satisfy a gauge-invariant uniqueness theorem and with the intriguing properties of the quotient $\mathcal{N}\mathcal{O}^r_X$ which does.

To discuss the key new idea we introduce in this paper, we first observe that amenability considerations imply that $\mathcal{NO}_X$ will not, in general, satisfy a gauge-invariant uniqueness theorem. Specifically, suppose that (G, P) is a quasi-lattice ordered group such that G is not amenable, but every finite subset of P has a supremum under the quasi-lattice order (finite-type Artin groups provide examples of this situation). Define a product system over P by letting $X_p = \mathbb{C}$ for each p. Then $\mathcal{NO}_X$ is isomorphic to the group C^*-algebra $C^*(G)$, and the quotient map from $C^*(G)$ to $C^*_r(G)$ preserves the gauge coaction and is injective on the coefficient algebra but is not injective. It is the *reduced* group C^*-algebra which satisfies a gauge-invariant uniqueness theorem, but this algebra lacks a universal property to induce homomorphisms to which this theorem may be applied.

Since it is the gauge-invariant uniqueness property we are interested in, we seek an analogue of $C^*_r(G)$ to the context of C^*-algebras associated to product systems. We desire an "intrinsic" definition of our C^*-algebra which, like the universal properties of other generalisations of Cuntz-Krieger algebras, gives us an effective tool for analysis. To this end, our C^*-algebra $\mathcal{NO}^r_X$ is described in terms of a co-universal property. Specifically, in Section 4 we prove that for X in a large class of product systems there exists a unique C^*-algebra $\mathcal{NO}^r_X$ which: (1) is generated by an injective Nica covariant Toeplitz representation of X; (2) carries a coaction of G compatible with the canonical gauge coaction on Fowler's $\mathcal{T}_{\mathrm{cov}}(X)$; and (3) has the property that given any other C^*-algebra B generated by an injective Nica covariant representation ψ of X and carrying a coaction β of G compatible with the gauge coaction, there is a canonical homomorphism $\phi\colon B \to \mathcal{NO}^r_X$. We also establish that this homomorphism ϕ is injective if and only if ψ is Cuntz-Nica-Pimsner covariant and β is normal.

We identify amenability hypotheses which imply that the canonical coaction on $\mathcal{NO}_X$ is normal. It then follows from our main theorem that $\mathcal{NO}_X$ and $\mathcal{NO}^r_X$ coincide under the same amenability hypotheses. From this we obtain a gauge-invariant uniqueness property of the usual form for $\mathcal{NO}_X$, see Corollary 4.11.

The basic idea of a co-universal property of a C^*-algebra has appeared before, notably in Exel's work on Fell bundles [14] which we use in our analysis, in Katsura's work [22] on C^*-algebras associated to a single bimodule which our work generalises, and in the work of Laca and Crisp-Laca on Toeplitz algebras and their boundary quotients [6, 24]. However, this article is, to our knowledge, the first time that the co-universal property has been used as the defining property of a C^*-algebra. In Section 5 we make extensive — and to our knowledge quite novel — use of the defining co-universal property of $\mathcal{NO}^r_X$ to prove that in various special cases $\mathcal{NO}^r_X$ is isomorphic to appropriate reduced crossed-product like C^*-algebras. In particular, we feel justified in regarding the algebras $\mathcal{NO}^r_X$ and $\mathcal{NO}_X$ as reduced- and full twisted crossed products of the algebra A by a generalised partial action of the group G.

We wish to emphasise the power and utility of the co-universal property of $\mathcal{NO}^r_X$. In particular, the co-universal property involves only the defining relations of the Nica-Toeplitz algebra and not the Cuntz-Pimsner covariance condition introduced in [35]. Since this last is a very technical relation, and difficult to check in practice, it is a significant advantage of our approach that we do not need to check it in any of our applications. In each case, we instead check that the algebra A which we wish to compare with $\mathcal{NO}_X$ is generated by an injective Nica covariant Toeplitz representation of X, use

the co-universal property to obtain a surjective homomorphism $\phi\colon A \to \mathcal{NO}_X^r$, and then use properties of A to prove that ϕ is injective. Particularly interesting is that when A is some sort of reduced crossed product, we then obtain, almost for free, isomorphism of the corresponding full crossed product with $\mathcal{NO}_X$. The point is that proving directly, for example, the isomorphism between $\mathcal{NO}_X$ and the full crossed product associated with Crisp and Laca's boundary quotient algebra would require using the universal properties in both directions, and hence checking both the Cuntz-Pimsner relation, and Crisp and Laca's elementary relations associated with the essential spectrum of the quasi-lattice ordered group. The effort could not be reduced by application of a gauge-invariant uniqueness theorem in either direction because such a theorem only applies when the full and reduced C^*-algebras coincide.

The results of the paper are organised in three main sections following a preliminaries section. In Section 3, we analyse the fixed-point algebra in $\mathcal{T}_{\mathrm{cov}}(X)$ and establish, for a large class of product systems, that any representation of $\mathcal{NO}_X$ which is injective on the coefficient algebra is injective on the whole fixed-point algebra. This answers a question of Sims and Yeend [35]. In Section 4, we define $\mathcal{NO}_X^r$ and prove our uniqueness theorems for it. Using Exel's results, we also establish a gauge-invariant uniqueness theorem for $\mathcal{NO}_X$ under appropriate amenability hypotheses. Finally, in Section 5 we use our theorems, most notably the co-universal property, to establish for each of a variety of reduced crossed-product algebras A an isomorphism of A with $\mathcal{NO}_X^r$ for an appropriate product system X. We also prove in Section 5 that Katsura's construction of a Hilbert bimodule from a topological graph yields, for each compactly aligned topological higher-rank graph Λ in the sense of Yeend, a compactly aligned product system X over $\mathbb{N}^k$ of Hilbert bimodules, and that for this X, $\mathcal{NO}_X$ is isomorphic to the C^*-algebra of Yeend's boundary-path groupoid of Λ (see [37]). We have included an appendix detailing how and when coactions descend to quotients, and when the resulting coaction is normal. These results are surely known, but were difficult to locate in the literature, at least in the specific context of full coactions with which we deal in this paper.

Towards the late stages of completing this paper, the second named author learned of the possible connection between our work and that of Arveson in [3]. It seems that the existence of a co-universal algebra for our systems could probably be derived from Arveson's results. Since Arveson's algebras are not obtained constructively we believe that our explicit construction and identification of the co-universal algebra is of independent interest. We thank Hangfeng Li for pointing us to [3].

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2. Preliminaries

For a discrete group G we write $g \mapsto i_G(g)$ for the canonical inclusion of G as unitaries in the full group C^*-algebra $C^*(G)$. We let λ_G denote the left regular representation

of G. We denote by A a C^*-algebra. An unadorned tensor product of C^*-algebras will denote the minimal tensor product.

Much of what follows is a summary of [35, Section 2]. We refer the reader to [4, 25, 33] for more detail.

2.1. **Hilbert bimodules.** A right-Hilbert A-module is a complex vector space X endowed with a right A-module structure and an A-valued A-sesquilinear form $\langle \cdot\, , \cdot \rangle_A$ (*-linear in the first variable) such that X is complete in the norm $\|x\|_A := \|\langle x, x\rangle_A\|^{1/2}$. A map $T\colon X \to X$ is said to be *adjointable* if there is a map $T^*\colon X \to X$ such that $\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A$ for all $x, y \in X$. Every adjointable operator on X is norm-bounded and linear, and the adjoint T^* is unique. The collection $\mathcal{L}(X)$ of adjointable operators on X endowed with the operator norm is a C^*-algebra. The ideal of generalised compact operators $\mathcal{K}(X) \lhd \mathcal{L}(X)$ is the closed span of the operators $x \otimes y^*\colon z \mapsto x \cdot \langle y, z\rangle_A$ where x and y range over X.

A *right-Hilbert A–A bimodule* is a right-Hilbert A module X endowed with a left action of A by adjointable operators, which we formalise as a homomorphism $\phi\colon A \to \mathcal{L}(X)$. Each C^*-algebra A is a right-Hilbert A–A bimodule ${}_AA_A$ with actions given by multiplication and inner product given by $(a, b) \mapsto a^*b$. The homomorphism that takes $a \in A$ to left-multiplication by a on ${}_AA_A$ is an isomorphism of A onto $\mathcal{K}({}_AA_A)$.

The balanced tensor product $X \otimes_A Y$ of right-Hilbert A–A bimodules X and Y (with left actions ϕ and ρ) is the completion of the vector space spanned by elements $x \otimes_A y$ where $x \in X$ and $y \in Y$ subject to the relation $x \cdot a \otimes_A y = x \otimes_A \rho(a)y$, in the norm determined by the inner-product

$$\langle x_1 \otimes_A y_1, x_2 \otimes_A y_2\rangle_A = \langle y_1, \langle x_1, x_2\rangle_A \cdot y_2\rangle_A.$$

There is a right action of A on $X \otimes_A Y$ given by $(x \otimes_A y) \cdot a = x \otimes_A (y \cdot a)$. With this $X \otimes_A Y$ is a right-Hilbert A-module. For $S \in \mathcal{L}(X)$, the formula $(S \otimes 1_{\mathcal{L}(Y)})(x \otimes_A y) = Sx \otimes y$ determines an adjointable operator on $X \otimes_A Y$. In particular, there is a left action of A on $X \otimes_A Y$ implemented by the homomorphism $a \mapsto \phi(a) \otimes 1_{\mathcal{L}(Y)}$. With this, $X \otimes_A Y$ is a right-Hilbert A–A bimodule.

2.2. **Semigroups and product systems of Hilbert bimodules.** A product system over a unital, discrete semigroup P consists of a semigroup X equipped with a semigroup homomorphism $d\colon X \to P$ such that $X_p := d^{-1}(p)$ is a right-Hilbert A–A bimodule for each $p \in P$, $X_e = {}_AA_A$, and the multiplication on X implements isomorphisms $X_p \otimes_A X_q \cong X_{pq}$ for $p, q \in P \setminus \{e\}$ and the right and left actions of $X_e = A$ on each X_p. For $p \in P$, we denote the homomorphism of A to $\mathcal{L}(X_p)$ which implements the left action by ϕ_p. We automatically have $\phi_{pq}(a)(xy) = (\phi_p(a)x)y$ for all $x \in X_p$, $y \in X_q$ and $a \in A$.

Given $p, q \in P$ with $p \neq e$, there is a homomorphism $\iota_p^{pq}\colon \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ characterised by

$$\iota_p^{pq}(S)(xy) = (Sx)y \text{ for all } x \in X_p,\ y \in X_q \text{ and } S \in \mathcal{L}(X_p). \tag{2.1}$$

Identifying $\mathcal{K}(X_e)$ with A as above, one can also define $\iota_e^q\colon \mathcal{K}(X_e) \to \mathcal{L}(X_q)$ simply by letting $\iota_e^q = \phi_q$ for all q, see [35, §2.2].

We will be interested in semigroups arising in quasi-lattice ordered groups in the sense of Nica [26]. Given a discrete group G and a subsemigroup P of G such that

$P \cap P^{-1} = \{e\}$, we say that (G, P) is a *quasi-lattice ordered group* if, under the partial order $g \leq h \iff g^{-1}h \in P$, any two elements p, q in G with a common upper bound in P have a least common upper bound $p \vee q$ in P (it follows from [5, Lemma 7] that this definition is equivalent to Nica's original definition from [26], which Fowler also uses in [18], and to the definition Crisp and Laca use in [5] and [6]). We write $p \vee q = \infty$ to indicate that $p, q \in G$ have no common upper bound in P, and we write $p \vee q < \infty$ otherwise. As is standard, see [26], if $p \vee q < \infty$ for all $p, q \in P$, we say that P is *directed*.

Given a quasi-lattice ordered group (G, P), a product system X over P is called *compactly aligned* (as in [18, Definition 5.7]) if $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$ whenever $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, and $p \vee q < \infty$. An explanation is in order here: Fowler only defines compactly aligned in the case that each X_p is essential as a left A-module. However, since we use $\iota_p^{p\vee q}(S)$ and not $S \otimes_A 1_{p^{-1}(p\vee q)}$ as in [18, Definition 5.7], and since these make sense also when $p = e$, we can work with compactly aligned product systems of not necessarily essential bimodules.

2.3. **Representations of product systems.** Given a product system X over P, a Toeplitz representation ψ of X in a C^*-algebra B is a map $\psi : X \to B$ such that:

(1) for each p, $\psi_p := \psi|_{X_p} \colon X_p \to B$ is linear and ψ_e is a homomorphism;
(2) ψ takes multiplication in X to multiplication in B; and
(3) $\psi_e(\langle x, y\rangle_A^p) = \psi_p(x)^*\psi_p(y)$ for all $x, y \in X_p$ (where $\langle x, y\rangle_A^p$ denotes the A-valued inner product on X_p).

In particular, each ψ_p is a Toeplitz representation of X_p in B, see [18]. A Toeplitz representation ψ of X is *injective* provided that the homomorphism $\psi_e : X_e \to B$ is injective. Note that property (3) then implies that ψ_p is an isometry for each $p \in P$. In this paper, we will frequently drop the word *Toeplitz* and refer to a map ψ as above simply as a representation of X.

Given a Toeplitz representation ψ of a product system X, there are $*$-homomorphisms $\psi^{(p)} \colon \mathcal{K}(X_p) \to B$ such that $\psi^{(p)}(x \otimes y^*) = \psi_p(x)\psi_p(y)^*$ for all $x, y \in X_p$ (see for example [27]). Proposition 2.8 of [18] shows that there is a universal C^*-algebra $\mathcal{T}_X$ generated by a universal Toeplitz representation i of X.

Now suppose that (G, P) is a quasi-lattice ordered group and X is a compactly aligned product system over P. We say that a Toeplitz representation ψ of X is *Nica covariant* if

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise} \end{cases}$$

for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$ (see also [18, Definition 5.7]). Let $\mathcal{T}_{\text{cov}}(X)$ be the quotient of $\mathcal{T}_X$ by the ideal generated by the elements

$$i^{(p)}(S)i^{(q)}(T) - i^{(p\vee q)}(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T))$$

where $p, q \in P$, $S \in \mathcal{K}(X_p)$, $T \in \mathcal{K}(X_q)$, and by convention, $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) = 0$ if $p \vee q = \infty$. The composition of the quotient map from $\mathcal{T}_X$ onto $\mathcal{T}_{\text{cov}}(X)$ with i is a Nica covariant Toeplitz representation $i_X \colon X \to \mathcal{T}_{\text{cov}}(X)$ with the following universal property: if ψ is a Nica covariant Toeplitz representation of X in B there is a $*$-homomorphism $\psi_* : \mathcal{T}_{\text{cov}}(X) \to B$ such that $\psi_* \circ i_X = \psi$. Thus if X is a compactly aligned product system

of essential Hilbert bimodules, then $\mathcal{T}_{\mathrm{cov}}(X)$ coincides with Fowler's algebra (denoted by the same symbol) from [18] defined for not necessarily compactly aligned product systems of essential Hilbert bimodules. By an argument similar to [18, Theorem 6.3] we have

$$\mathcal{T}_{\mathrm{cov}}(X) = \overline{\operatorname{span}}\,\{\, i_X(x)i_X(y)^* \mid x, y \in X \,\}. \tag{2.2}$$

It follows from (2.2) that if the image of a Nica covariant Toeplitz representation ψ of X generates B as a C^*-algebra, then $B = \overline{\operatorname{span}}\,\{\, \psi(x)\psi(y)^* \mid x, y \in X \,\}$.

2.4. **The algebra $\mathcal{NO}_X$.** To define Cuntz-Pimsner covariance of representations, we must first summarise some definitions from [35, Section 3]. We say that a predicate statement $\mathcal{P}(s)$ (where $s \in P$) is true for large s if for every $q \in P$ there exists $r \geq q$ such that $\mathcal{P}(s)$ is true whenever $r \leq s$.

Assume (G, P) is quasi-lattice ordered and X is a compactly aligned product system over P. Define $I_e = A$, and for each $q \in P \setminus \{e\}$ write $I_q := \bigcap_{e<p\leq q} \ker(\phi_p)$. We then write $\widetilde{X}_q$ for the right-Hilbert A–A bimodule

$$\widetilde{X}_q := \bigoplus\nolimits_{p\leq q} X_p \cdot I_{p^{-1}q}.$$

The homomorphism implementing the left action is denoted $\tilde{\phi}_q$. We say that X is *$\tilde{\phi}$-injective* if the homomorphisms $\tilde{\phi}_q$ are all injective.

For $p \not\leq q \in P$ we define $\iota_p^q(T) = 0_{\mathcal{L}(X_q)}$ for all $T \in \mathcal{L}(X_p)$. Recalling the definitions of the maps ι_p^{pq} from Section 2.2, we then have homomorphisms $\tilde{\iota}_p^q \colon \mathcal{L}(X_p) \to \mathcal{L}(\widetilde{X}_q)$ for all $p, q \in P$ with $p \neq e$ defined by $\tilde{\iota}_p^q(T) = \bigoplus_{r\leq q} \iota_p^r(T)$ for all $p, q \in P$ with $p \neq e$. When $p = e$, similar to the above there is a homomorphism $\tilde{\iota}_e^q : \mathcal{K}(X_e) \to \mathcal{L}(\widetilde{X}_q)$.

Suppose that X is $\tilde{\phi}$-injective. We say that a Nica covariant Toeplitz representation ψ of X in a C^*-algebra B is *Cuntz-Nica-Pimsner covariant* (or *CNP-covariant*) if it has the following property:

> $\sum_{p\in F} \psi^{(p)}(T_p) = 0_B$ whenever $F \subset P$ is finite, $T_p \in \mathcal{K}(X_p)$ for each $p \in F$, and $\sum_{p\in F} \tilde{\iota}_p^q(T_p) = 0$ for large q.

As in [35, Proposition 3.12], if X is $\tilde{\phi}$-injective, we write $\mathcal{NO}_X$ for the universal C^*-algebra generated by a CNP-covariant representation j_X of X, and call it the Cuntz-Nica-Pimsner algebra of X. By [35, Remark 4.2], the hypothesis that X is $\tilde{\phi}$-injective ensures that j_X is an injective representation. By [35, Theorem 4.1], X is $\tilde{\phi}$-injective (and hence $\mathcal{NO}_X$ is defined and j_X is an injective representation) whenever each ϕ_p is injective, and also whenever each bounded subset of P has a maximal element.

3. Analysis of the core

In this section we lay the foundation for the proof of our main result Theorem 4.1. To do this we shall analyse the fixed-point algebra of $\mathcal{T}_{\mathrm{cov}}(X)$ under a canonical coaction δ. As a corollary we show that under certain conditions $\mathcal{NO}_X$ satisfies criterion (B) of [35, Section 1]. Throughout the rest of the article, we write $q_{\mathrm{CNP}} \colon \mathcal{T}_{\mathrm{cov}}(X) \to \mathcal{NO}_X$ for the canonical surjection arising from the universal property of $\mathcal{T}_{\mathrm{cov}}(X)$.

Lemma 3.1. *Let (G, P) be a quasi-lattice ordered group and let X be a product system over P of right-Hilbert A–A bimodules. Let $\psi\colon X \to B$ be a Toeplitz representation of X. Then:*

1. *If $p \le t \in P$, $T \in \mathcal{K}(X_p)$, and $x \in X_t$, then $\psi_t(\iota_p^t(T)(x)) = \psi^{(p)}(T)\psi_t(x)$;*
2. *If $t < r \le s \in P$, $T \in \mathcal{K}(X_r)$, and $x \cdot a \in X_t \cdot I_{t^{-1}s}$, then $\psi^{(r)}(T)\psi_t(x \cdot a) = 0$.*

Proof. (1) If $p = e$ then (1) follows from the observations that $\mathcal{K}(X_e) \cong A$ and $\iota_e^t := \phi_t$, so suppose $p \ne e$. Since $\operatorname{span}\{ xy \mid x \in X_p, y \in X_{p^{-1}t} \}$ is dense in X_t, and since $\operatorname{span}\{ w \otimes z^* \mid w, z \in X_p \}$ is dense in $\mathcal{K}(X_p)$, to prove (1) it suffices to show that for $x, w, z \in X_p$ and $y \in X_{p^{-1}t}$ we have

$$\psi_t(\iota_p^t(w \otimes z^*)(xy)) = \psi^{(p)}(w \otimes z^*)\psi_t(xy).$$

Using (2.1), we calculate:

$$\begin{aligned}
\psi_t(\iota_p^t(w \otimes z^*)(xy)) &= \psi_p(w \cdot \langle z, x\rangle_A^p)\psi_{p^{-1}t}(y) \\
&= \psi_p(w)\psi_e(\langle z, x\rangle_A^p)\psi_{p^{-1}t}(y) \\
&= \psi_p(w)\psi_p^*(z)\psi_p(x)\psi_{p^{-1}t}(y) \\
&= \psi^{(p)}(w \otimes z^*)\psi_t(xy)
\end{aligned}$$

as required in (1).

(2) If $t = e$ then $x \cdot a \in I_s$, so $x \cdot a \in \ker(\phi_r)$. By using that $\mathcal{K}(X_r) = \overline{\operatorname{span}}\,\{y \otimes z^* : x, y \in X_r\}$ one easily checks that $\psi_e(b)\psi^{(r)}(S) = \psi^{(r)}(\phi_r(b)S)$ for $b \in X_r$ and $S \in \mathcal{K}(X_r)$. By taking adjoints and letting $b = (x \cdot a)^*$ and $S = T^*$, it follows that

$$\psi^{(r)}(T)\psi_e(x \cdot a) = \psi^{(r)}(T\phi_r(x \cdot a)) = 0.$$

Now suppose $t \ne e$. Fix $y \in X_t$, $z \in X_{t^{-1}r}$, and $v \in X_r$. It suffices to show that $\psi^{(r)}(v \otimes (yz)^*)\psi_t(x \cdot a) = 0$. Since $a \in I_{t^{-1}s} = \bigcap_{e<q\le t^{-1}s} \ker(\phi_q)$, we have $\phi_{t^{-1}r}(a) = 0$, and hence

$$\begin{aligned}
\psi^{(r)}(v \otimes (yz)^*)\psi_t(x \cdot a) &= \psi_r(v)\psi_{t^{-1}r}(z)^*\psi_t(y)^*\psi_t(x \cdot a) \\
&= \psi_r(v)\psi_{t^{-1}r}(z)^*\psi_e(\langle y, x \cdot a\rangle_A^t) \\
&= \psi_r(v)\psi_{t^{-1}r}(z)^*\psi_e(\langle y, x\rangle_A^t a) \\
&= \psi_r(v)(\psi_e(a^*\langle x, y\rangle_A^t)\psi_{t^{-1}r}(z))^* \\
&= \psi_r(v)\psi_{t^{-1}r}(\phi_{t^{-1}r}(a^*\langle x, y\rangle_A^t)z)^* \\
&= \psi_r(v)\psi_{t^{-1}r}(\phi_{t^{-1}r}(a)^*\phi_{t^{-1}r}(\langle x, y\rangle_A^t)z)^* \\
&= 0.
\end{aligned}$$

□

Lemma 3.1(2) says, roughly, if $r \in tP \setminus \{t\}$, then $\psi^{(r)}(T) \in B$ annihilates $\psi_t(X_t \cdot I_{t^{-1}s})$ whenever $s \in rP$. The next corollary says that when X is compactly aligned and ψ is Nica covariant, we can replace the requirement that $r \in tP \setminus \{t\}$ with the much weaker requirement that $t \notin rP$, and that s is a common upper bound for t and r.

Corollary 3.2. *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Let $\psi\colon X \to B$ be a Nica covariant representation of X. Suppose $p, t \le s \in P$ and $p \not\le t$. Then for $T \in \mathcal{K}(X_p)$ and $x \cdot a \in X_t \cdot I_{t^{-1}s}$, we have $\psi^{(p)}(T)\psi_t(x \cdot a) = 0$.*

Proof. Let $(E_k)_{k\in K}$ be an approximate identity for $\mathcal{K}(X_t \cdot I_{t^{-1}s})$. Since $p, t \leq s$, we have $p \vee t < \infty$. Hence Nica covariance and the fact that each $E_k \in \mathcal{K}(X_t)$ imply that

$$\begin{aligned}\psi^{(p)}(T)\psi_t(x \cdot a) &= \lim_{k\in K} \psi^{(p)}(T)\psi_t(E_k(x \cdot a)) \\ &= \lim_{k\in K} \psi^{(p)}(T)\psi^{(t)}(E_k)\psi_t(x \cdot a) \\ &= \lim_{k\in K} \psi^{(p\vee t)}\big(\iota_p^{p\vee t}(T)\iota_t^{p\vee t}(E_k)\big)\psi_t(x \cdot a).\end{aligned}$$

Since $p \not\leq t$ forces $t < p \vee t$ and since $p \vee t \leq s$, the result now follows from statement (2) of Lemma 3.1. $\square$

Lemma 3.3. *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is by injective homomorphisms, or that P is directed. Let $\psi\colon X \to B$ be an injective Nica covariant representation of X. Fix a finite subset $F \subset P$ and fix operators $T_p \in \mathcal{K}(X_p)$ for each $p \in F$ satisfying $\sum_{p\in F} \psi^{(p)}(T_p) = 0$. Then $\sum_{p\in F} \tilde{\iota}_p^s(T_p) = 0$ for large s.*

Proof. Fix $q \in P$. We must show that there exists $r \geq q$ such that for every $s \geq r$, we have $\sum_{p\in F} \tilde{\iota}_p^s(T_p) = 0_{\mathcal{L}(\widetilde{X}_s)}$.

List the elements of F as $p_1, \ldots, p_{|F|}$. Define $r_0 := q$, and inductively, for $1 \leq i \leq |F|$, define

$$r_i := \begin{cases} r_{i-1} \vee p_i & \text{if } r_{i-1} \vee p_i < \infty \\ r_{i-1} & \text{otherwise.} \end{cases}$$

Set $r := r_{|F|}$, and note that this satisfies $r \geq q$. With no extra assumptions on the quasi-lattice ordered group we also have $r \geq p$ whenever $p \in F$ satisfies $r \vee p < \infty$. If P is directed then $r = q \vee \big(\bigvee_{p\in F} p\big)$, and is an upper bound for F.

Let $s \geq r$. To show that $\sum_{p\in F} \tilde{\iota}_p^s(T_p) = \sum_{p\in F} \big(\bigoplus_{t\leq s} \iota_p^t(T_p)\big)$ is equal to the zero operator on $\widetilde{X}_s = \bigoplus_{t\leq s} X_t \cdot I_{t^{-1}s}$ we shall prove that $\sum_{p\in F, p\leq t} \iota_p^t(T_p)|_{X_t\cdot I_{t^{-1}s}} = 0_{\mathcal{L}(X_t\cdot I_{t^{-1}s})}$ for each $t \leq s$. Indeed, for $x \cdot a \in X_t \cdot I_{t^{-1}s}$, using Lemma 3.1(1) we have

$$\begin{aligned}\psi_t\big(\textstyle\sum_{p\in F, p\leq t} \iota_p^t(T_p)(x \cdot a)\big) &= \textstyle\sum_{p\in F, p\leq t} \psi_t(\iota_p^t(T_p)(x \cdot a)) \\ &= \textstyle\sum_{p\in F, p\leq t} \psi^{(p)}(T_p)\psi_t(x \cdot a).\end{aligned} \tag{3.1}$$

We claim that (3.1) is equal to $\sum_{p\in F} \psi^{(p)}(T_p)\psi_t(x \cdot a)$. We will establish this claim under each of the additional hypotheses of the lemma. Note that the claim comes down to proving

$$\psi^{(p)}(T_p)\psi_t(x \cdot a) = 0 \text{ if } p \in F, p \not\leq t. \tag{3.2}$$

Suppose that the ϕ_p are injective. Then $I_{t^{-1}s} = 0$ for $t < s$, and so $a = 0$ in (3.2) unless $t = s$. Thus it suffices in this case to show that $\psi^{(p)}(T_p)\psi_s(x) = 0$ for $T_p \in \mathcal{K}(X_p)$ and $x \in X_s$. By choice of r and s and the assumption $p \not\leq s$, we necessarily have $p \vee s = \infty$. Let $(E_k)_{k\in K}$ be an approximate identity for $\mathcal{K}(X_s)$. By Nica covariance, $\psi^{(p)}(T_p)\psi_s(x) = \lim_{k\in K} \psi^{(p)}(T_p)\psi^{(s)}(E_k)\psi_s(x) = 0$.

Now suppose that P is directed. Then $p \leq r \leq s$ and equation (3.2) follows from Corollary 3.2.

Thus we have in both cases that $\psi_t(\sum_{p\in F, p\leq t} \iota_p^t(T_p)(x\cdot a)) = \sum_{p\in F} \psi^{(p)}(T_p)\psi_t(x\cdot a)$. Since this last sum is equal to 0 by hypothesis, and since the representation ψ is injective, so that in particular every ψ_t is injective, it follows that $\sum_{p\in F, p\leq t} \iota_p^t(T_p)(x\cdot a) = 0$, as needed. □

Remark 3.4. The hypotheses that either the left actions are all injective, or P is directed are genuinely necessary in Lemma 3.3; see Example 3.9.

Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. It follows (see [18, Proposition 5.10]) from the Nica-covariance of i_X that

$$\mathcal{F} := \overline{\operatorname{span}}\,\{\, i_X(x)i_X(y)^* \mid x, y \in X, d(x) = d(y) \,\} \tag{3.3}$$

is closed under multiplication, and thus that it is a C^*-subalgebra of $\mathcal{T}_{\text{cov}}(X)$. We call this subalgebra the *core* of $\mathcal{T}_{\text{cov}}(X)$.

For any discrete group G there is a homomorphism $\delta_G \colon C^*(G) \to C^*(G) \otimes C^*(G)$ given by $\delta_G(g) = i_G(g) \otimes i_G(g)$. Recall that a full coaction of G on a C^*-algebra A is an injective homomorphism $\delta \colon A \to A \otimes C^*(G)$ which is nondegenerate (in the sense that $\overline{\operatorname{span}}\, \delta(A)(A \otimes C^*(G)) = A \otimes C^*(G)$) and satisfies the coaction identity $(\delta \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta$ (see, for example, [29]. All coactions in this paper are full. The generalised fixed-point algebra of A with respect to δ is $A_e^\delta := \{a \in A \mid \delta(a) = a \otimes i_G(e)\}$.

We will now show that there is a coaction of G on $\mathcal{T}_{\text{cov}}(X)$ whose generalised fixed-point algebra is equal to the core $\mathcal{F}$. For Fowler's $\mathcal{T}_{\text{cov}}(X)$ associated to a not-necessarily compactly aligned product system over P of essential A–A bimodules (where (G, P) is quasi-lattice ordered), Proposition 4.7 in [18] and the discussion preceding [18, Theorem 6.3] imply the existence of a coaction with similar properties as δ in the next result. We present a different and more direct proof here.

Proposition 3.5. *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Then there is a coaction δ of G on $\mathcal{T}_{\text{cov}}(X)$ such that $\delta(i_X(x)) = i_X(x) \otimes i_G(d(x))$ for all $x \in X$.*

Proof. Let $\psi : X \to \mathcal{T}_{\text{cov}}(X) \otimes C^*(G)$ be the map $x \mapsto i_X(x) \otimes i_G(d(x))$. It is straightforward to check that ψ is a Nica covariant representation of X. It follows from the universal property of $\mathcal{T}_{\text{cov}}(X)$ that there is a $*$-homomorphism $\delta : \mathcal{T}_{\text{cov}}(X) \to \mathcal{T}_{\text{cov}}(X) \otimes C^*(G)$ such that $\delta(i_X(x)) = \psi(x) = i_X(x) \otimes i_G(d(x))$ for all $x \in X$. We will show that δ is a coaction.

We first show that δ is nondegenerate. Let $(\theta_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for $\mathcal{F}$. We claim that $(\theta_\lambda)_{\lambda\in\Lambda}$ is also an approximate identity for $\mathcal{T}_{\text{cov}}(X)$. Since $\mathcal{T}_{\text{cov}}(X)$ is the closure of the span of elements of the form $i_X(x)i_X(y)^*$, it suffices to show that $\theta_\lambda i_X(x)i_X(y)^* \to i_X(x)i_X(y)^*$ for all $x, y \in X$. Fix $x, y \in X$ and let $p = d(x) \in P$. By [33, Proposition 2.31] we may write $x = z \cdot \langle z, z\rangle_A^p = (z \otimes z^*)(z)$ for some $z \in X$, and then $i_X(x)i_X(y)^* = i_X^{(p)}(z \otimes z^*)i_X(z)i_X(y)^*$. Since $i_X^{(p)}(z \otimes z^*) \in \mathcal{F}$, we have $\theta_\lambda i_X^{(p)}(z \otimes z^*) \to i_X^{(p)}(z \otimes z^*)$, and hence $\theta_\lambda i_X(x)i_X(y)^* \to i_X(x)i_X(y)^*$ as claimed. Since $\delta(\theta_\lambda) = \theta_\lambda \otimes 1$ for each $\lambda \in \Lambda$, the approximate identity $(\theta_\lambda)_{\lambda\in\Lambda}$ is mapped under δ to an approximate identity for $\mathcal{T}_{\text{cov}}(X) \otimes C^*(G)$, and it follows that $\overline{\operatorname{span}}\, \delta(\mathcal{T}_{\text{cov}}(X))(\mathcal{T}_{\text{cov}}(X) \otimes C^*(G)) =$

$\mathcal{T}_{\mathrm{cov}}(X) \otimes C^*(G)$. By checking on generators, it is easy to see that δ satisfies the coaction identity $(\delta \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_{\mathcal{T}_{\mathrm{cov}}(X)} \otimes \delta_G) \circ \delta$, and δ is injective since $\mathrm{id}_{\mathcal{T}_{\mathrm{cov}}(X)} = (\mathrm{id}_{\mathcal{T}_{\mathrm{cov}}(X)} \otimes \epsilon) \circ \delta$ where $\epsilon : C^*(G) \to \mathbb{C}$ is the integrated form of the representation $g \mapsto 1$. □

We call the above coaction δ of G on $\mathcal{T}_{\mathrm{cov}}(X)$ for the *gauge coaction* on $\mathcal{T}_{\mathrm{cov}}(X)$. It follows from equation (2.2) that the generalised fixed-point algebra $\mathcal{T}_{\mathrm{cov}}(X)^\delta_e = \{\, a \in \mathcal{T}_{\mathrm{cov}}(X) \mid \delta(a) = a \otimes i_G(e) \,\}$ is equal to the core $\mathcal{F}$.

Since $i_X^{(p)} \colon \mathcal{K}(X_p) \to \mathcal{T}_X$ satisfies $i_X^{(p)}(x \otimes y^*) = i_X(x) i_X(y)^*$ and each $\mathcal{K}(X_p) = \overline{\operatorname{span}}\,\{\, x \otimes y^* \mid x, y \in X_p \,\}$ by definition, we have

$$\mathcal{F} = \overline{\operatorname{span}}\,\{\, i_X^{(p)}(T) \mid p \in P \text{ and } T \in \mathcal{K}(X_p) \,\}. \tag{3.4}$$

We say that a subset F of P is $\vee$-*closed* if, whenever $p, q \in F$ satisfy $p \vee q < \infty$, we have $p \vee q \in F$. Let $\mathcal{P}^{\vee}_{\mathrm{fin}}(P)$ denote the set of finite $\vee$-closed subsets of P; then $\mathcal{P}^{\vee}_{\mathrm{fin}}(P)$ is directed under set inclusion (see [18, p. 367]). If $F \in \mathcal{P}^{\vee}_{\mathrm{fin}}(P)$ is bounded, then $\bigvee_{p \in F} p$ is a maximal element in F.

For $p \in P$, we write B_p for the C^*-subalgebra $i_X^{(p)}(\mathcal{K}(X_p)) \subset \mathcal{T}_{\mathrm{cov}}(X)$. For each finite $\vee$-closed subset F of P, we denote by B_F the linear subspace

$$B_F := \textstyle\sum_{p \in F} B_p = \Big\{ \textstyle\sum_{p \in F} i_X^{(p)}(T_p) \mid T_p \in \mathcal{K}(X_p) \text{ for each } p \in F \Big\} \subset \mathcal{T}_{\mathrm{cov}}(X). \tag{3.5}$$

Equation (3.4) implies that

$$\mathcal{F} = \overline{\textstyle\bigcup_{F \in \mathcal{P}^{\vee}_{\mathrm{fin}}(P)} B_F}. \tag{3.6}$$

Lemma 3.6. *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. For each finite $\vee$-closed subset F of P, the space B_F is a C^*-subalgebra of $\mathcal{F}$.*

Proof. Fix a finite $\vee$-closed subset F of P. Then B_F is a subspace of $\mathcal{F}$ by definition. One can check on spanning elements that it is closed under adjoints and multiplication (for the latter, one uses the Nica covariance of the universal representation i_X of X in $\mathcal{T}_{\mathrm{cov}}(X)$). It therefore suffices to show that B_F is norm-closed.

We proceed by induction on $|F|$. If $|F| = 1$, then $F = \{p\}$ for some $p \in P$, and then $B_F = B_p = i_X^{(p)}(\mathcal{K}(X_p))$ is the range of a C^*-homomorphism and hence closed.

Now suppose that B_F is closed whenever $|F| \le k$. Suppose that $F \subset P$ is $\vee$-closed with $|F| = k+1$. Since F is finite, we may fix an element m of F which is minimal in the sense that for $p \in F \setminus \{m\}$, we have $p \not\le m$. The sets $\{m\}$ and $F \setminus \{m\}$ are both finite and $\vee$-closed, and it follows from our induction hypothesis that B_m and $B_{F\setminus\{m\}}$ are C^*-subalgebras of $\mathcal{F}$. For $p \in F \setminus \{m\}$, we have $p \not\le m$ by choice of m and it follows that if $p \vee m < \infty$, then $p \vee m \in F \setminus \{m\}$. Hence for $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_m)$, we have

$$i_X^{(p)}(S) i_X^{(m)}(T) = i_X^{(p\vee m)}(\iota_p^{p\vee m}(S) \iota_m^{p\vee m}(T)) \in i_X^{(p\vee m)}(\mathcal{K}(X_{p\vee m})) \subset B_{F\setminus\{m\}}.$$

Similarly, $i_X^{(m)}(T) i_X^{(p)}(S) \in B_{F\setminus\{m\}}$, so by linearity, $ab, ba \in B_{F\setminus\{m\}}$ for all $a \in B_{F\setminus\{m\}}$ and $b \in B_m$. Corollary 1.8.4 of [10] now shows that $B_F = B_m + B_{F\setminus\{m\}}$ is norm closed. □

The following proposition is the key technical result which we will use in the proof of our main theorem in the next section.

Proposition 3.7. *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is by injective homomorphisms, or that P is directed. Let $\psi\colon X \to B$ be an injective Nica covariant representation of X and let $\psi_* : \mathcal{T}_{\mathrm{cov}}(X) \to B$ be the homomorphism characterised by $\psi = \psi_* \circ i_X$. Then $\ker(\psi_*) \cap \mathcal{F} \subset \ker(q_{\mathrm{CNP}})$.*

Proof. By [1, Lemma 1.3], equation (3.6) and Lemma 3.6, it suffices to show that $\ker(\psi_*) \cap B_F \subset \ker(q_{\mathrm{CNP}})$ for each $F \in \mathcal{P}^{\vee}_{\mathrm{fin}}(P)$. For this, we fix $F \in \mathcal{P}^{\vee}_{\mathrm{fin}}(P)$ and generalised compact operators $T_p \in \mathcal{K}(X_p)$ for $p \in F$, so that $c := \sum_{p\in F} i_X^{(p)}(T_p)$ is a typical element of B_F. Suppose that $c \in \ker(\psi_*)$; we must show that $c \in \ker(q_{\mathrm{CNP}})$ as well. Since the representation ψ is injective, Lemma 3.3 implies that $\sum_{p\in F} \tilde{\iota}^s_p(T_p) = 0$ for large s in the sense of [35, Definition 3.8]. Since j_X is CNP-covariant, it follows that

$$q_{\mathrm{CNP}}(c) = \textstyle\sum_{p\in F} j_X^{(p)}(T_p) = 0$$

as well, so $c \in \ker(q_{\mathrm{CNP}})$ as required. □

We now have enough machinery to confirm that $\mathcal{NO}_X$ indeed satifies criterion (B) of [35, Section 1] when the left actions on the fibres of X are all injective, or P is directed. One could use the following theorem to prove directly a gauge-invariant uniqueness theorem for $\mathcal{NO}_X$ when G is amenable, but since this will be an easy corollary of our more general main result, we will not pursue this line of attack. Recall from [35] that $\mathcal{NO}_X$ has the following universal property: for each CNP-covariant representation ψ of X there is a homomorphism $\Pi\psi$ such that $\Pi\psi \circ j_X = \psi$.

Theorem 3.8. *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Assume either that the left actions on the fibres of X are all injective, or that P is directed and X is $\tilde{\phi}$-injective. Let $\psi\colon X \to B$ be a CNP-covariant representation of X in a C^*-algebra B. Then the induced homomorphism $\Pi\psi : \mathcal{NO}_X \to B$ is injective on $q_{\mathrm{CNP}}(\mathcal{F})$ if and only if ψ is injective as a Toeplitz representation.*

Proof. Suppose that $\Pi\psi$ is injective on $q_{\mathrm{CNP}}(\mathcal{F})$. By [35, Theorem4.1], j_X is injective on A. Hence $\psi_e = \Pi\psi \circ (j_X)_e$ is also injective, and thus ψ is an injective Toeplitz representation.

Now suppose that ψ is injective as a Toeplitz representation; we must show that $\Pi\psi$ is injective on $q_{\mathrm{CNP}}(\mathcal{F})$. By definition of $\mathcal{NO}_X$ and of $\Pi\psi$, we have $\Pi\psi \circ q_{\mathrm{CNP}} = \psi_*$. Proposition 3.7 therefore implies that $\ker(\Pi\psi \circ q_{\mathrm{CNP}}) \cap \mathcal{F} \subset \ker(q_{\mathrm{CNP}})$. Hence $\Pi\psi$ is injective on $q_{\mathrm{CNP}}(\mathcal{F})$ as claimed. □

Example 3.9. We present an example of a product system X in which the left actions are not injective, and P is not directed, and the conclusion of Lemma 3.3 fails. It is easy to see that the conclusions of Proposition 3.7 and Theorem 3.8 both fail in this example (see also Remark 4.2).

Let the quasi-lattice ordered group be $(G, P) = (\mathbb{F}_2, \mathbb{F}_2^+)$, and denote by a and b the generators of $\mathbb{F}_2^+$. Define a product system over $\mathbb{F}_2^+$ by $X_{a^n} = \mathbb{C}$ for $n \in \mathbb{N}$ and $X_p = 0$ for all other elements of $\mathbb{F}_2^+$. This is compactly aligned since $\mathcal{L}(X_p) = \mathcal{K}(X_p)$ for each p, but the left actions are not all injective (and $a \vee b = \infty$). Define $\psi\colon X \to \mathbb{C}$ by $\psi_p(x) = x$ for $x \in X_p$ and $p \in \mathbb{F}_2^+$. Then ψ is an injective Nica covariant Toeplitz representation of X.

Let 1_p be the identity in $\mathcal{L}(X_p)$ for each p, and note that $1_e \in \mathcal{K}(X_e)$ and $1_a \in \mathcal{K}(X_a)$. We have that $\psi^{(e)}(1_e) = \psi^{(a)}(1_a) = 1$, so $\psi^{(e)}(1_e) - \psi^{(a)}(1_a) = 0$. However, we claim that $\tilde{\iota}^e_s(1_e) - \tilde{\iota}^a_s(1_a)$ is not equal to 0 for large s. Indeed, note that

$$I_q = \begin{cases} 0 & \text{if } q = a^n \text{ for some } n \in \mathbb{N} \\ \mathbb{C} & \text{otherwise} \end{cases}$$

for $q \in P \setminus \{e\}$. It follows that if $q \geq b$, then $X_e \cdot I_q = X_e$, and so

$$(\iota^e_e(1_e) - \iota^e_a(1_a))|_{X_e \cdot I_q} = \iota^e_e(1_e) - \iota^e_a(1_a) = \iota^e_e(1_e) \neq 0,$$

which shows that $\tilde{\iota}^e_s(1_e) - \tilde{\iota}^a_s(1_a) \neq 0$ for all $s \geq b$.

4. The co-universal C^*-algebra and the uniqueness theorems

We begin this section with our main theorem. Before stating it, we introduce some terminology: given a quasi-lattice ordered group (G, P) and a product system X over P of right-Hilbert A–A bimodules, a Toeplitz representation $\psi\colon X \to B$ is *gauge-compatible* if there is a coaction β of G on B such that

$$\beta(\psi(x)) = \psi(x) \otimes i_G(d(x)) \quad \text{for all } x \in X. \tag{4.1}$$

Suppose that $\psi_1 : X \to B_1$ and $\psi_2 : X \to B_2$ are two gauge-compatible Toeplitz representations of X, that β_i is a coaction of G on B_i satisfying $\beta_i(\psi_i(x)) = \psi_i(x) \otimes i_G(d(x))$ for all $x \in X$ and $i = 1, 2$, and that $\phi : B_1 \to B_2$ is a $*$-homomorphism satisfying $\phi \circ \psi_1 = \psi_2$. Then ϕ is equivariant for β_1 and β_2, meaning that $(\phi \otimes \mathrm{id}_{C^*(G)}) \circ \beta_1 = \beta_2 \circ \phi$.

Since our main result depends on the technical hypothesis that X is $\tilde{\phi}$-injective, we emphasise that the results of [35] imply that this is automatic whenever either the left actions on the fibres of X are all injective or every bounded subset of P has a maximal element.

Theorem 4.1. *Let (G, P) be a quasi-lattice ordered group and X a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is injective, or that P is directed and X is $\tilde{\phi}$-injective. Then there exists a triple $(\mathcal{NO}^r_X, j^r_X, \nu^n)$ which is co-universal for gauge-compatible injective Nica covariant representations of X in the following sense:*

1. *$\mathcal{NO}^r_X$ is a C^*-algebra, j^r_X is an injective Nica covariant representation of X whose image generates $\mathcal{NO}^r_X$, and ν^n is a coaction of G on $\mathcal{NO}^r_X$ such that $\nu^n(j^r_X(x)) = j^r_X(x) \otimes i_G(d(x))$ for all $x \in X$.*
2. *If $\psi\colon X \to B$ is an injective Nica covariant gauge-compatible representation whose image generates B then there is a surjective $*$-homomorphism $\phi\colon B \to \mathcal{NO}^r_X$ such that $\phi(\psi(x)) = j^r_X(x)$ for all $x \in X$.*

Moreover, the representation j^r_X is CNP-covariant, the coaction ν^n is normal, and $(\mathcal{NO}^r_X, j^r_X, \nu^n)$ is the unique triple satisfying (1) *and* (2)*: if (C, ρ, γ) satisfies the same two conditions, then there is an isomorphism $\phi\colon C \to \mathcal{NO}^r_X$ such that $j^r_X = \phi \circ \rho$ and ϕ is equivariant for γ and ν^n.*

Remark 4.2. Although Example 3.9 does not satisfy the assumptions of Theorem 3.8, it nevertheless does admit a co-universal algebra as described in Theorem 4.1; but this co-universal algebra is a proper quotient of the algebra $\mathcal{NO}^r_X$ that we shall construct

later. Specifically, it is not difficult to see that every Toeplitz representation of the system X described in Example 3.9 is automatically Nica covariant, and that there is a bijective correspondence between Toeplitz representations of X and Toeplitz representations of X_a which takes injective representations to injective representations and gauge-compatible representations to gauge-compatible representations. It thus follows that both the Toeplitz algebra and the covariant Toeplitz algebra of X are equal to the classical Toeplitz algebra $\mathcal{T}$ (generated by a single isometry), and that $C(\mathbb{T})$ has the co-universal property described in Theorem 4.1 with respect to the system X. Moreover, one can check that the Toeplitz representation of X into $\mathcal{T}$ is CNP-covariant, and it thus follows that $\mathcal{NO}_X$, and hence also the $\mathcal{NO}_X^r$ which we will construct later, are both isomorphic to $\mathcal{T}$ and not to $C(\mathbb{T})$.

To prove Theorem 4.1, we first need to recall a few facts about Fell bundles and their C^*-algebras, and about coactions. The main reference to Fell bundles and properties of the full cross-sectional algebra of a bundle is [16, Section VIII.17.2]. For the relationship between topologically graded C^*-algebras and C^*-algebras associated to Fell bundles, in particular the reduced C^*-algebra of a bundle, we refer to [14]. The connection between discrete coactions and Fell bundles was explored in [29]. In [29, Definition 3.5], Quigg introduced a reduced C^*-algebra of a Fell bundle together with a coaction. The subtle point that the reduced constructions from [14] and [29] are compatible (although far from obviously so) was clarified in [13, page 749].

Notation 4.3. Suppose that δ is a coaction of a discrete group G on a C^*-algebra A. For every $g \in G$, let $A_g^\delta := \{ a \in A \mid \delta(a) = a \otimes i_G(g) \}$ be the spectral subspace of A at g. By [29], the disjoint union of the spectral subspaces $A_g^\delta \times \{g\}$ for $g \in G$ forms a Fell bundle over G, which we call *the Fell bundle associated to* δ (see [13, page 748]).

Conversely, if $(\mathcal{A}, G)$ is a Fell bundle then it follows from [29, Proposition 3.3] that there is a canonical coaction $\delta_{\mathcal{A}}$ on the full cross-sectional algebra $C^*(\mathcal{A})$ such that $\delta_{\mathcal{A}}(a_g) = a_g \otimes i_G(g)$ for all a_g in the fiber of $\mathcal{A}$ over g and all g in G.

If $(\mathcal{A}, G)$ is a Fell bundle over G and A a cross-sectional algebra of $(\mathcal{A}, G)$ (in the sense that A is a C^*-completion of the algebra of finitely supported sections on $\mathcal{A}$), then we say that A is topologically graded if there exists a contractive conditional expectation from A to A_e which vanishes on A_g for each $g \in G \setminus \{e\}$ (see [14, Definition 3.4]). The *reduced cross-sectional algebra* $C_r^*(\mathcal{A})$ defined in [14] was shown to be minimal among topologically graded cross-sectional algebras A. To be more precise, if A is any topologically graded cross-sectional algebra of $(\mathcal{A}, G)$, [14, Theorem 3.3] shows that there exists a surjective homomorphism $\lambda_{\mathcal{A}} : A \to C_r^*(\mathcal{A})$ such that $\lambda_{\mathcal{A}} \circ \eta_{\mathcal{A}} = \kappa_{\mathcal{A}}$ where $\eta_{\mathcal{A}}$ and $\kappa_{\mathcal{A}}$ are the embeddings of the algebra of finitely supported sections on $\mathcal{A}$ into A and $C_r^*(\mathcal{A})$, respectively. On the other hand, the universal property of $C^*(\mathcal{A})$ (see [16, VIII.16.11]) gives a surjective homomorphism

$$\phi_{\mathcal{A}} \colon C^*(\mathcal{A}) \to A \tag{4.2}$$

such that $\phi_{\mathcal{A}} \circ \gamma_{\mathcal{A}} = \eta_{\mathcal{A}}$ where $\gamma_{\mathcal{A}}$ is the embedding of the algebra of finitely supported sections on $\mathcal{A}$ into $C^*(\mathcal{A})$.

If δ is a coaction of G on A and $(\mathcal{A}, G)$ is the Fell bundle associated to δ, it follows from [29, Lemma 1.3] (see also [13, page 749]) that A is a topologically graded cross-sectional algebra of $\mathcal{A}$. We shall adopt the notation A^r for the reduced cross-sectional

algebra of the bundle $(\mathcal{A}, G)$ arising from the coaction δ on A. (We choose not to cram δ into the notation A^r for the sake of readability: the coaction δ will always be clear from context.) By the considerations of the previous paragraph applied to A and $C^*(\mathcal{A})$, there are surjective homomorphisms

$$\lambda_{\mathcal{A}}\colon A \to A^r \text{ and } \Lambda_{\mathcal{A}}\colon C^*(\mathcal{A}) \to A^r \tag{4.3}$$

such that $\lambda_{\mathcal{A}} \circ \eta_{\mathcal{A}} = \kappa_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}} \circ \gamma_{\mathcal{A}} = \kappa_{\mathcal{A}}$; and hence $\lambda_{\mathcal{A}} \circ \phi_{\mathcal{A}} = \Lambda_{\mathcal{A}}$ (see, for example, [13]).

As explained in [13, page 749], A^r (defined by its minimality, or co-universal property) is the same as the reduced algebra from [29] associated to $(\mathcal{A}, G)$. By [29, Definition 3.5], there exists a coaction δ^n (the n stands for "normal"; see Remark 4.4) on A^r with the property that

$$\delta^n\big(\lambda_{\mathcal{A}}(a_g)\big) = \lambda_{\mathcal{A}}(a_g) \otimes i_G(g) \text{ for all } a_g \in A^\delta_g. \tag{4.4}$$

Recall that a coaction η of G on a C^*-algebra C is called *normal* if $(\mathrm{id} \otimes \lambda_G) \otimes \eta$ is injective. Every coaction η of G on C has a *normalisation*: the quotient C^n of C by $\ker((\mathrm{id} \otimes \lambda_G) \otimes \eta)$ carries a coaction $\tilde{\eta}$ which is automatically normal. In our set-up, A^r is isomorphic to A^n, and this isomorphism identifies the coaction δ^n defined by (4.4) with the normalisation $\tilde{\delta}$ of δ (see [13, Lemma 2.1]). Moreover, δ^n may also be identified with the normalisation of $\delta_{\mathcal{A}}$ by construction (see [29]). In particular, as the notation suggests, δ^n is a normal coaction on A^r.

Remark 4.4. Our choice of notation (A^r, δ^n) for the system obtained above from (A, δ) may seem a little perverse when either (A^r, δ^r) or (A^n, δ^n) would at least be internally consistent. We have our reasons. The notation A^r is, for us, much more appealing than A^n for two reasons: firstly, it coincides with our key reference [14]; and secondly, there is strong evidence that the object obtained in this way from the algebra $\mathcal{NO}_X$ of [35] should be regarded as a reduced crossed product (see Section 5). However, the notation δ^r would be a most unfortunate choice because it suggests a reduced coaction (that is, one taking values in $A \otimes C^*_r(G)$) whereas we have been careful to use only full coactions throughout this paper for the sake of consistency and self-containment; in particular, δ^n is a full normal coaction. See the notation after [29, Definition 3.5] for a similar point of view.

Remark 4.5. Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system of Hilbert bimodules over P. By Proposition 3.5, $\mathcal{T}_{\mathrm{cov}}(X)$ admits a coaction δ, and hence gives rise to a Fell bundle $\mathcal{B} = (\mathcal{T}_{\mathrm{cov}}(X)^\delta_g \times \{g\})_{g \in G}$ over G. The generalised fixed-point algebra $\mathcal{T}_{\mathrm{cov}}(X)^\delta_e$ is precisely the algebra $\mathcal{F}$ of (3.3).

Let ι denote the map from $\bigcup_{g \in G} \mathcal{T}_{\mathrm{cov}}(X)^\delta_g$ to the algebra of finitely supported sections on $\mathcal{B}$ such that the restriction of ι to each $\mathcal{T}_{\mathrm{cov}}(X)^\delta_g$, identified with $\mathcal{T}_{\mathrm{cov}}(X)^\delta_g \times \{g\}$, is the canonical embedding of $\mathcal{T}_{\mathrm{cov}}(X)^\delta_g \times \{g\}$. We then have that $\eta_{\mathcal{B}} \circ \iota$ is the inclusion of $\bigcup_{g \in G} \mathcal{T}_{\mathrm{cov}}(X)^\delta_g$ into $\mathcal{T}_{\mathrm{cov}}(X)$. We claim that the map $\gamma_{\mathcal{B}} \circ \iota \circ i_X : X \to C^*(\mathcal{B})$ is a Nica covariant Toeplitz representation. Indeed, to check this we use that i_X is a Nica covariant representation and that $\gamma_{\mathcal{B}} \circ \iota$ is compatible with the multiplication and involution and restricts to a linear map on each fiber $\mathcal{T}_{\mathrm{cov}}(X)^\delta_g$ and to a $*$-homomorphism on the fiber $\mathcal{T}_{\mathrm{cov}}(X)^\delta_e$. Then the universal property of $\mathcal{T}_{\mathrm{cov}}(X)$ supplies a $*$-homomorphism

$\zeta : \mathcal{T}_{\mathrm{cov}}(X) \to C^*(\mathcal{B})$ such that $\zeta \circ i_X = \gamma_{\mathcal{B}} \circ \iota \circ i_X$. By (4.2), there is a surjective homomorphism $\phi_{\mathcal{B}} \colon C^*(\mathcal{B}) \to \mathcal{T}_{\mathrm{cov}}(X)$ such that $\phi_{\mathcal{B}} \circ \gamma_{\mathcal{B}} = \eta_{\mathcal{B}}$. We then have

$$\phi_{\mathcal{B}}(\zeta(i_X(x))) = \phi_{\mathcal{B}}(\gamma_{\mathcal{B}}(\iota(i_X(x)))) = \eta_{\mathcal{B}}(\iota(i_X(x))) = i_X(x)$$

and

$$\zeta(\phi_{\mathcal{B}}(\gamma_{\mathcal{B}}(\iota(i_X(x)))) = \zeta(\eta_{\mathcal{B}}(\iota(i_X(x)))) = \zeta(i_X(x)) = \gamma_{\mathcal{B}}(\iota(i_X(x)))$$

for each $x \in X$, from which it follows that ζ is the inverse of $\phi_{\mathcal{B}}$. Hence $\phi_{\mathcal{B}}$ is an isomorphism from $C^*(\mathcal{B})$ to $\mathcal{T}_{\mathrm{cov}}(X)$ which is equivariant for $\delta_{\mathcal{B}}$ and δ.

Suppose that X is $\tilde{\phi}$-injective. Let $\mathcal{NO}_X$ be the Cuntz-Nica-Pimsner algebra of X and j_X the universal CNP-covariant representation. By the proof of [35, Proposition 3.12] and (3.6), the kernel of the canonical homomorphism $q_{\mathrm{CNP}} \colon \mathcal{T}_{\mathrm{cov}}(X) \to \mathcal{NO}_X$ is generated by its intersection with $\mathcal{F}$. Therefore Proposition A.1 applied to the coaction δ on $\mathcal{T}_{\mathrm{cov}}(X)$ yields a *gauge coaction* ν on $\mathcal{NO}_X$. The spectral subspaces

$$(\mathcal{NO}_X)^\nu_g := \{\, c \in \mathcal{NO}_X \mid \nu(c) = c \otimes i_G(g) \,\}$$

give rise to a Fell bundle $\mathcal{N}$, and it follows as above from the universal property of $\mathcal{NO}_X$ (see [35, Proposition 3.12]) that $\phi_{\mathcal{N}} \colon C^*(\mathcal{N}) \to \mathcal{NO}_X$ is an isomorphism which is equivariant for $\delta_{\mathcal{N}}$ and ν.

Proof of Theorem 4.1. Let $\mathcal{NO}^r_X$ be the reduced cross-sectional algebra of the Fell bundle $\mathcal{N}$, and $\lambda_{\mathcal{N}} \colon \mathcal{NO}_X \to \mathcal{NO}^r_X$ the homomorphism from (4.3). Put $j^r_X := \lambda_{\mathcal{N}} \circ j_X$, and define ν^n to be the normal coaction on $\mathcal{NO}^r_X$ described in (4.4).

To prove property (1), note that $\mathcal{NO}_X$ is generated by the injective CNP-covariant representation j_X, see [35, Proposition 3.12 and Theorem 4.1]. So j^r_X is CNP-covariant. Since $\lambda_{\mathcal{N}}$ is surjective, $\mathcal{NO}^r_X$ is generated by j^r_X. Further, $\lambda_{\mathcal{N}}$ restricts to a bijection from $(\mathcal{NO}_X)^\nu_e$ to $(\mathcal{NO}^r_X)^{\nu^n}_e$, and since $j_X(A) \subset (\mathcal{NO}_X)^\nu_e$ the representation j^r_X is injective. Finally it follows from (4.4) that $\nu^n(j^r_X(x)) = j^r_X(x) \otimes i_G(d(x))$ for all $x \in X$.

We next show that $(\mathcal{NO}^r_X, j^r_X)$ has property (2). Suppose $\psi \colon X \to B$ is as in (2), and let β be a coaction on B such that (4.1) holds. For $g \in G$, let $B^\beta_g = \{\, b \in B \mid \beta(b) = b \otimes i_G(g) \,\}$. Then [29, Lemma 1.3 and 1.5] and [14, Theorem 3.3] imply that $\{B^\beta_g\}_{g\in G}$ is a topological grading of B. The universal property of $\mathcal{T}_{\mathrm{cov}}(X)$ gives a $*$-homomorphism $\psi_* \colon \mathcal{T}_{\mathrm{cov}}(X) \to B$ such that $\psi = \psi_* \circ i_X$. Since the image of ψ generates B, $\psi_*(\mathcal{F}) = B^\beta_e$, and so $I_0 := \psi_*(\ker(q_{\mathrm{CNP}})) \cap B^\beta_e = \psi_*(\ker(q_{\mathrm{CNP}}) \cap \mathcal{F})$ is an ideal of B^β_e. Let I be the ideal of B generated by I_0.

By construction, I is an induced ideal in the sense of [14, Definition 3.10]. Let $\pi \colon B \to B/I$ be the quotient map. By [14, Proposition 3.11], $\{\pi(B^\beta_g)\}_{g\in G}$ is a topological grading of B/I.

Since the image of ψ generates B, we have $\pi(\psi_*(\mathcal{T}_{\mathrm{cov}}(X)^\delta_g)) = \pi(B^\beta_g)$ for all $g \in G$. We aim to show that for every $g \in G$ we have

$$\ker(\pi \circ \psi_*) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_g = \ker(q_{\mathrm{CNP}}) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_g. \tag{4.5}$$

It will then follow that the two Fell bundles $\tilde{\mathcal{B}} := (\pi(B^\beta_g) \times \{g\})_{g\in G}$ and $((\mathcal{NO}_X)^\nu_g \times \{g\})_{g\in G}$ are isometrically isomorphic. Indeed, (4.5) implies that for every $g \in G$ there is an isomorphism φ_g from $(\mathcal{NO}_X)^\nu_g$ onto $\pi(B^\beta_g)$ given by $\varphi_{d(x)}(q_{CNP}(i_X(x))) = \pi \circ \psi_*(i_X(x))$ for all $x \in X$. These isomorphisms are compatible with the Fell bundle structure, in the sense that: $\varphi_{g_1}(c_1)\varphi_{g_2}(c_2) = \varphi_{g_1 g_2}(c_1 c_2)$ for $g_1, g_2 \in G$, $c_1 \in (\mathcal{NO}_X)^\nu_{g_1}$

and $c_2 \in (\mathcal{NO}_X)^\nu_{g_2}$; and $\varphi_g(c)^* = \varphi_{g^{-1}}(c^*)$ for $g \in G$ and $c \in (\mathcal{NO}_X)^\nu_g$. Hence the isomorphisms φ_g induce the claimed isomorphism between $\tilde{\mathcal{B}}$ and $\mathcal{N}$. Thus every cross sectional algebra of $\tilde{\mathcal{B}}$ is also a cross sectional algebra of $\mathcal{N}$ and vice versa, and the co-universal properties of $\mathcal{NO}^r_X$ and $C^*_r(\tilde{\mathcal{B}})$ then imply that there is an isomorphism $\tilde{\varphi}\colon \mathcal{NO}^r_X \to C^*_r(\tilde{\mathcal{B}})$ such that $\lambda_{\tilde{\mathcal{B}}} \circ \varphi_g$ and $\tilde{\varphi} \circ \lambda_{\mathcal{N}}$ agree on $(\mathcal{NO}_X)^\nu_g$ for all g.

Let $\phi := (\tilde{\varphi})^{-1} \circ \lambda_{\tilde{\mathcal{B}}} \circ \pi$. Then ϕ is a homomorphism from B onto $\mathcal{NO}^r_X$, and we have

$$\begin{aligned}\phi(\psi(x)) = (\tilde{\varphi})^{-1}(\lambda_{\tilde{\mathcal{B}}}(\pi(\psi_*(i_X(x))))) &= (\tilde{\varphi})^{-1}(\lambda_{\tilde{\mathcal{B}}}(\varphi_{d(x)}(q_{\mathrm{CNP}}(i_X(x)))))\\ &= \lambda_{\mathcal{N}}(q_{\mathrm{CNP}}(i_X(x))) = \lambda_{\mathcal{N}}(j_X(x)) = j^r_X(x),\end{aligned}$$

for all $x \in X$, as claimed.

We first prove (4.5) when $g = e$. So we claim that $\ker(\pi \circ \psi_*) \cap \mathcal{F} = \ker(q_{\mathrm{CNP}}) \cap \mathcal{F}$. If $c \in \ker(q_{\mathrm{CNP}}) \cap \mathcal{F}$, then $\psi_*(c) \in I_0 \subset I$, proving the right to left inclusion. To prove the other inclusion, note that since $\pi \circ \psi_* = (\pi \circ \psi)_*$, it suffices by Proposition 3.7 to show that the Toeplitz representation $\pi \circ \psi$ is injective.

Fix $a \in A$ with $\pi(\psi(a)) = 0$. Then $\psi(a) \in I \cap B^\beta_e$. Since $\psi_*(\ker(q_{\mathrm{CNP}}))$ is an ideal, it contains I, and therefore $I \cap B^\beta_e \subset \psi_*(\ker(q_{\mathrm{CNP}})) \cap B^\beta_e = I_0$. It follows that there exists a $y \in \ker(q_{\mathrm{CNP}}) \cap \mathcal{F}$ such that $\psi_*(y) = \psi(a)$. Hence $y - i_X(a) \in \ker(\psi_*) \cap \mathcal{F}$. Since $\ker(\psi_*) \cap \mathcal{F} \subset \ker(q_{\mathrm{CNP}})$ by Proposition 3.7 applied to ψ, it follows that $i_X(a) \in \ker(q_{\mathrm{CNP}}) \cap i_X(A)$. However, $\ker(q_{\mathrm{CNP}}) \cap i_X(A) = \{0\}$, hence $i_X(a) = 0$, and therefore $a = 0$. This proves that the representation $\pi \circ \psi$ is injective, and thus that $\ker(\pi \circ \psi_*) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_e = \ker(q_{\mathrm{CNP}}) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_e$.

Now let g be an arbitrary element of G. Then

$$\begin{aligned}c \in \ker(\pi \circ \psi_*) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_g \iff c^*c &\in \ker(\pi \circ \psi_*) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_e = \ker(q_{\mathrm{CNP}}) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_e,\\ \iff c &\in \ker(q_{\mathrm{CNP}}) \cap \mathcal{T}_{\mathrm{cov}}(X)^\delta_g.\end{aligned}$$

Hence (2) is established.

Finally, for the uniqueness assertion, suppose that (C, ρ, γ) also satisfies (1) and (2). Then property (2) for $\mathcal{NO}^r_X$ gives a homomorphism ϕ from C to $\mathcal{NO}^r_X$, the corresponding property for C gives a homomorphism from $\mathcal{NO}^r_X$ to C, and these two homomorphisms are mutually inverse. □

We saw in Remark 4.5 that $\mathcal{NO}_X$ is isomorphic to the full cross sectional algebra $C^*(\mathcal{N})$ of its associated Fell bundle arising from the gauge coaction ν. This fact has interesting implications. To explain this point, we need to recall some terminology from [12]. First, a coaction η of a group G on C is maximal if the canonical map from the iterated coaction crossed product $C \times_\eta G \times_{\hat{\eta}} G$ to $C \otimes \mathcal{K}(l^2)$ is an isomorphism. Second, a maximal coaction system (D, G, ϵ) is a maximalisation of (C, G, η) if there is an equivariant surjective homomorphism from D to C which induces an isomorphism of the coaction crossed products $D \times_\epsilon G$ and $C \times_\eta G$. Then [12, Proposition 4.2] implies that the canonical coaction $\nu_{\mathcal{N}}$ on $C^*(\mathcal{N})$ is a maximalisation of ν, in fact the unique one with the same underlying Fell bundle; moreover $\mathcal{NO}_X \cong C^*(\mathcal{N})$ means precisely that ν itself is maximal. At the other extreme, a system (D, G, ϵ) is called a normalisation of (C, G, η) if the coaction ϵ is normal and there is an equivariant surjective homomorphism from C to D which induces an isomorphism of the coaction crossed products $C \times_\eta G$ and $D \times_\epsilon G$.

Corollary 4.6. *Assume the hypotheses of Theorem 4.1. Let $\psi : X \to B$ be an injective CNP-covariant representation of X which is gauge-compatible for a coaction β on B, and such that ψ generates B. Then the following hold.*

(a) *The coaction system $(\mathcal{NO}_X, G, \nu)$ is a maximalisation of (B, G, β).*
(b) *The coaction system $(\mathcal{NO}^r_X, G, \nu^n)$ is a normalisation of (B, G, β).*

Proof. The universal property of $\mathcal{NO}_X$ gives a surjective homomorphism $\Pi\psi : \mathcal{NO}_X \to B$ which is equivariant for ν and β. Theorem 4.1 gives a homomorphism $\phi : B \to \mathcal{NO}^r_X$ which is equivariant for β and ν^n. We have $\phi \circ \Pi\psi = \lambda_{\mathcal{N}}$. Then it follows as in the proof of [13, Lemma 2.1] that the induced map $\lambda_{\mathcal{N}} \times G$ is an isomorphism from $\mathcal{NO}_X \times_\nu G$ onto $\mathcal{NO}^r_X \times_{\nu^n} G$ and satisfies $\lambda_{\mathcal{N}} \times G = (\phi \times G) \circ (\Pi\psi \times G)$. Therefore $\Pi\psi \times G$ is also an isomorphism, which shows that ν is a maximalisation of β, as claimed in (a). Likewise, $\phi \times G$ becomes an isomorphism from $B \times_\beta G$ onto $\mathcal{NO}^r_X \times_{\nu^n} G$, so (b) follows. □

The main reason for proceeding to a gauge-invariant uniqueness theorem for $\mathcal{NO}_X$ via Theorem 4.1 rather than proving it directly from Theorem 3.8 is that we feel that the co-universal property as a defining property of $\mathcal{NO}^r_X$ is just as important as — and in some ways more natural than — the defining universal property of $\mathcal{NO}_X$.

In particular when $\mathcal{NO}_X$ and $\mathcal{NO}^r_X$ coincide, the definition as a co-universal C^*-algebra has the advantage over the definition as a universal C^*-algebra that it involves only the natural defining relations for $\mathcal{T}_{\mathrm{cov}}(X)$ which are present in the Fock representation, and does not involve the complicated (and difficult to check) Cuntz-Pimsner covariance condition. This has clear advantages when trying to establish Cuntz-Nica-Pimsner algebras as models for other classes of examples (see Remark 5.7). Moreover, when $\mathcal{NO}_X$ and $\mathcal{NO}^r_X$ do not coincide, it is unclear what makes $\mathcal{NO}_X$ worthy of singling out for study beyond the bare fact that it is defined by a universal property with respect to a relation which holds in the co-universal C^*-algebra.

Example 4.7. If the advantage of the definition of $\mathcal{NO}^r_X$ as a co-universal algebra is that it bypasses the troublesome Cuntz-Pimsner covariance condition, then a natural next question is whether or not $\mathcal{NO}^r_X$ is actually co-universal for injective (not necessarily Nica covariant) gauge-compatible Toeplitz representations of X, thus allowing us to bypass the Nica covariance condition as well.

The answer in general is "No:" there exist product systems which admit no such co-universal C^*-algebra. To see this, let $(G, P) = (\mathbb{F}_2, \mathbb{F}_2^+)$ and let $X_p = \mathbb{C}$ for every $p \in P$. Then $\mathcal{L}(X_p) = \mathcal{K}(X_p)$ for all p, so X is compactly aligned, and all left actions are injective. Denote the generators of $\mathbb{F}_2$ by a and b. Suppose that (C, ρ) is a co-universal pair for injective gauge-compatible Toeplitz representations of X. Note that there is an injective Toeplitz representation ψ of X on $C^*_r(\mathbb{F}_2)$ determined by $\psi(1_p) := \lambda_{\mathbb{F}_2}(p)$. By [30, Example 1.15] or [28, Proposition 2.4], there is a (full) coaction $\delta_{\mathbb{F}_2}$ of $\mathbb{F}_2$ on $C^*_r(\mathbb{F}_2)$ such that $\delta_{\mathbb{F}_2}(\lambda_{\mathbb{F}_2}(g)) = \lambda_{\mathbb{F}_2}(g) \otimes i_{\mathbb{F}_2}(g)$ for all $g \in \mathbb{F}_2$. The integrated form of ψ is therefore gauge-compatible. By the co-universal property of (C, ρ), there is a surjective homomorphism $\phi \colon C^*_r(\mathbb{F}_2) \to C$ satisfying $\phi(\psi(1_p)) = \rho(1_p)$ for all $p \in P$. In particular, since $\lambda_{\mathbb{F}_2}(a)$ and $\lambda_{\mathbb{F}_2}(b)$ are unitaries, surjectivity of ϕ implies that

$$\rho(1_a)\rho(1_a)^*\rho(1_b)\rho(1_b)^* = \phi(\lambda_{\mathbb{F}_2}(a)\lambda_{\mathbb{F}_2}(a)^*\lambda_{\mathbb{F}_2}(b)\lambda_{\mathbb{F}_2}(b)^*) = \phi(1_{C^*_r(\mathbb{F}_2)}) = 1_C. \tag{4.6}$$

Since j_X^r is also an injective gauge-compatible Toeplitz representation of X, the co-universal property of (C,ρ) gives a surjective homomorphism $\eta\colon \mathcal{NO}_X^r \to C$ such that $\eta(j_X^r(1_p)) = \rho(1_p)$ for all $p \in \mathbb{F}_2$. Since j_X^r is Nica covariant, and since $a \vee b = \infty$ in $\mathbb{F}_2$, we have

$$\rho(1_a)\rho(1_a)^*\rho(1_b)\rho(1_b)^* = \eta(j_X^r(1_a)j_X^r(1_a)^*j_X^r(1_b)j_X^r(1_b)^*) = \eta(0) = 0. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain $1_C = 0$, so $C = \{0\}$, which contradicts the assumption that ρ is an injective representation of X.

The preceding example shows that a co-universal C^*-algebra for gauge-compatible injective Toeplitz representations need not exist. We will now show that if such a co-universal C^*-algebra does exist, then it must be isomorphic to $\mathcal{NO}_X^r$, and prove that it satisfies a kind of rudimentary gauge-invariant uniqueness theorem.

Corollary 4.8. *Let (G,P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is injective, or that P is directed and X is $\tilde{\phi}$-injective.*

1. *If $\phi\colon \mathcal{NO}_X^r \to B$ is a surjective $*$-homomorphism, then ϕ is injective if and only if $\phi|j_X^r(A)$ is injective and there is a coaction β of G on B such that $\beta \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \nu^n$.*
2. *Suppose that (C,ρ) is a co-universal pair for injective gauge-compatible (not necessarily Nica covariant) Toeplitz representations of X. Then there is a $*$-isomorphism $\phi\colon \mathcal{NO}_X^r \to C$ such that $\phi(j_X^r(x)) = \rho(x)$ for all $x \in X$.*

Proof. (1) The "only if" assertion is trivial. For the "if" assertion note that $\phi \circ j_X^r$ is an injective Nica covariant representation of X in B which is gauge-compatible, and whose image generates B. An application of Theorem 4.1 yields a surjection $B \to \mathcal{NO}_X^r$ which is an inverse for ϕ.

(2) Since $\mathcal{NO}_X^r$ is generated by an injective Toeplitz representation of X which is gauge-compatible, the co-universal property of (C,ρ) implies that there is a surjective homomorphism $\phi\colon \mathcal{NO}_X^r \to C$ such that $\phi(j_X^r(x)) = \rho(x)$ for all $x \in X$. Part (1) then implies that ϕ is injective and hence an isomorphism. □

Part (1) of the preceding corollary is used to prove statement (2), but is somewhat unsatisfactory as a stand alone result because there is no universal property to induce homomorphisms $\phi\colon \mathcal{NO}_X^r \to B$ of the desired form (compare with Definition 4.10 below). The following result is much more satisfactory in that it provides an injectivity criterion for the homomorphism $\phi\colon B \to \mathcal{NO}_X^r$ induced by Theorem 4.1(2).

Corollary 4.9. *Let (G,P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is injective, or that P is directed and X is $\tilde{\phi}$-injective. Let $\psi\colon X \to B$ be an injective Nica covariant gauge-compatible representation whose image generates B. Then the surjective $*$-homomorphism $\phi\colon B \to \mathcal{NO}_X^r$ of Theorem 4.1(2) is injective if and only if ψ is Cuntz-Pimsner covariant and β is normal.*

Proof. If ϕ is injective, then ψ is Cuntz-Nica-Pimsner covariant and β is normal because j_X^r and ν^n have these properties.

Now suppose that ψ is Cuntz-Nica-Pimsner covariant and β is normal. The universal property of $\mathcal{NO}_X$ gives a homomorphism $\Pi\psi\colon \mathcal{NO}_X \to B$. By definition, $\lambda_{\mathcal{N}}\colon \mathcal{NO}_X \to \mathcal{NO}^r_X$ satisfies $\lambda_{\mathcal{N}} = \phi \circ \Pi\psi$. The map $\lambda_{\mathcal{N}}$ restricts to an isomorphism of $(\mathcal{NO}_X)^\nu_e$ to $(\mathcal{NO}^r_X)^{\nu^n}_e$, hence ϕ restricts to an isomorphism $B^\beta_e \to (\mathcal{NO}^r_X)^{\nu^n}_e$. Since β is normal, it determines a faithful conditional expectation $\Phi^\beta\colon B \to B^\beta_e$. But ϕ intertwines Φ^β and the conditional expectation from $\mathcal{NO}^r_X$ to $(\mathcal{NO}^r_X)^{\nu^n}_e$, and so the standard argument implies that ϕ is injective. □

For the next corollary, we first define some additional terminology.

Definition 4.10. Fix a quasi-lattice ordered group (G, P) and a $\tilde{\phi}$-injective compactly aligned product system X over P. We say that $\mathcal{NO}_X$ has the *gauge-invariant uniqueness property* provided that the following is satisfied.

A surjective $*$-homomorphism $\phi : \mathcal{NO}_X \to B$ is injective if and only if:

(GI1) there is a coaction β of G on B such that $\beta \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \nu$, and
(GI2) the homomorphism $\phi|_{j_X(A)}$ is injective.

Corollary 4.11 (The gauge-invariant uniqueness theorem)**.** *Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is injective, or that P is directed and X is $\tilde{\phi}$-injective. The following are equivalent.*

1. *The coaction ν is normal.*
2. *The coaction ν^n is maximal.*
3. *The Fell bundle $\big((\mathcal{NO}_X)^\nu_g \times \{g\}\big)_{g\in G}$ is amenable.*
4. *The $*$-homomorphism $\lambda_{\mathcal{N}}\colon \mathcal{NO}_X \to \mathcal{NO}^r_X$ is an isomorphism.*
5. *$\mathcal{NO}_X$ has the gauge-invariant uniqueness property.*
6. *If $\psi_1 : X \to B_1$ and $\psi_2 : X \to B_2$ are two injective gauge-compatible CNP-covariant representations of X whose images generate B_1 and B_2 respectively, then there exists a $*$-isomorphism $\phi : B_1 \to B_2$ such that $\phi \circ \psi_1 = \psi_2$.*

Proof. That (1) and (4) are equivalent follows from the fact that ν^n is the normalisation of ν (see the last paragraph of Notation 4.3). The equivalence of (2) and (4) follows from [12, Proposition 4.2] and the fact, established in Remark 4.5, that $(\mathcal{NO}_X, G, \nu)$ is isomorphic to $(C^*(\mathcal{N}), G, \nu_{\mathcal{N}})$. The equivalence of (3) and (4) follows from the definition of amenability for Fell bundles in [14] and the fact that $\mathcal{NO}_X$ is isomorphic to $C^*(\mathcal{N})$. We will show that (4) implies (5), that (5) implies (6), and that (6) implies (4).

Suppose first that (4) holds. Let $\phi : \mathcal{NO}_X \to B$ be a surjective $*$-homomorphism. We must show that ϕ is injective if and only if conditions (GI1) and (GI2) hold. If ϕ is injective, then we may define β by $\beta := (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \nu \circ \phi^{-1}$, so condition (GI1) is satisfied. Moreover, [35, Theorem 4.1] implies that $(\phi \circ j_X)|_A$ is injective, so condition (GI2) is also satisfied. Now suppose that (GI1) and (GI2) hold. Then $\phi \circ j_X$ is a gauge-compatible Nica covariant representation whose image generates B, so by Theorem 4.1 there is a $*$-homomorphism $\tilde{\phi} : B \to \mathcal{NO}^r_X$ satisfying $\tilde{\phi} \circ (\phi \circ j_X) = j^r_X = \lambda_{\mathcal{N}} \circ j_X$. Since $\mathcal{NO}_X$ is generated by the elements $\{\, j_X(x) \mid x \in X \,\}$ and $\lambda_{\mathcal{N}}$ is an isomorphism, it follows that $(\lambda_{\mathcal{N}})^{-1} \circ \tilde{\phi}$ is an inverse for ϕ, and hence ϕ is injective as required.

Suppose that (5) holds. If $\psi : X \to B$ is an injective gauge-compatible CNP-covariant representation of X whose images generate B, then $\Pi\psi : \mathcal{NO}_X \to B$ is a surjective $*$-homomorphism such that $\Pi\psi \circ j_X = \psi$ and (GI1) and (GI2) are satisfied, hence is an isomorphism. Statement (6) follows.

Finally suppose (6) holds. It follows from [35, Proposition 3.12 and Theorem 4.1] and Theorem 4.1 that $j_X : X \to \mathcal{NO}_X$ and $j_X^r : X \to \mathcal{NO}_X^r$ are two injective gauge-compatible CNP-covariant representations whose images generate $\mathcal{NO}_X$ and $\mathcal{NO}_X^r$ respectively. Thus there exists a $*$-isomorphism $\phi : \mathcal{NO}_X^r \to \mathcal{NO}_X$ such that $\phi \circ j_X^r = j_X$. We then have that $\lambda_{\mathcal{N}} \circ \phi = \mathrm{id}_{\mathcal{NO}_X^r}$, from which (4) holds. □

It is of course interesting to know under which conditions $\mathcal{NO}_X$ has the gauge-invariant uniqueness property. Using Exel's work [14], we obtain the following conditions under which $\mathcal{NO}_X$ has the gauge-invariant uniqueness property.

Corollary 4.12. *Let (G, P) be a quasi-lattice ordered group and X a compactly aligned product system over P of right-Hilbert A–A bimodules. Suppose either that the left action on each fibre is injective, or that P is directed and X is $\tilde{\phi}$-injective. Then $\mathcal{NO}_X$ has the gauge-invariant uniqueness property in the following cases:*

1. *The group G is exact and the coaction δ of G on $\mathcal{T}_{\mathrm{cov}}(X)$ is normal.*
2. *The Fell bundle $\mathcal{B} = (\mathcal{T}_{\mathrm{cov}}(X)_g^\delta \times \{g\})_{g\in G}$ has the approximation property.*
3. *The Fell bundle $\mathcal{N} = \big((\mathcal{NO}_X)_g^\nu \times \{g\}\big)_{g\in G}$ has the approximation property.*
4. *The group G is amenable.*

Proof. Statement (1) follows from Corollary 4.11 because normality of δ implies normality of ν by Proposition A.5. For statement (2), let $J = \ker(q_{\mathrm{CNP}})$, let $\Phi^\delta \colon \mathcal{T}_{\mathrm{cov}}(X) \to \mathcal{T}_{\mathrm{cov}}(X)_e^\delta$ be the conditional expectation induced by the coaction δ, and let $\Phi^\nu \colon \mathcal{NO}_X \to (\mathcal{NO}_X^\nu)_e$ be the conditional expectation induced by the coaction ν. Then it follows from [14, Proposition 3.6] that $\ker(\lambda_{\mathcal{N}}) = \{\, b \in \mathcal{NO}_X \mid \Phi^\nu(b^*b) = 0 \,\}$ and thus that $\ker((j_X^r)_*) = q_{\mathrm{CNP}}^{-1}(\ker(\lambda_{\mathcal{N}})) = \{\, c \in \mathcal{T}_{\mathrm{cov}}(X) \mid \Phi^\delta(c^*c) \in J \,\}$. Hence, if the Fell bundle $\mathcal{B} = (\mathcal{T}_{\mathrm{cov}}(X)_g^\delta \times \{g\})_{g\in G}$ has the approximation property, then [14, Proposition 4.10] implies that $\ker((j_X^r)_*) = \ker(q_{\mathrm{CNP}})$. Thus $\lambda_{\mathcal{N}}$ is an isomorphism, and statement (2) then follows from Corollary 4.11. If the Fell bundle $\mathcal{N}$ has the approximation property, then it is amenable by [14, Theorem 4.6], hence (3) follows from Corollary 4.11. Finally, if G is amenable, then [14, Theorem 4.7] shows that the bundle $\mathcal{N}$ has the approximation property, so (4) follows from (3). □

Remark 4.13. Observe that the universal property of $\mathcal{NO}_X$ together with the co-universal property of $\mathcal{NO}_X^r$ stated in Theorem 4.1 imply that the canonical homomorphism from $\mathcal{NO}_X$ to $\mathcal{NO}_X^r$ factors through the image B of any injective CNP-covariant representation ψ of X which generates B and respects δ, hence ν. By Corollary 4.6, we see that when the gauge-invariant uniqueness theorem applies, it implies that the universal and co-universal algebras for gauge-compatible injective CNP-covariant representations of X agree, the gauge coaction is both normal and maximal, and all of the C^*-algebras B coincide.

Our motivating example was $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$, in which case condition (GI1) can be stated in the familiar terms of an action of the dual group $\mathbb{T}^k$.

Corollary 4.14. *Let X be a compactly aligned product system over $\mathbb{N}^k$. A surjective $*$-homomorphism $\phi : \mathcal{NO}_X \to B$ is injective if and only if:*

(1) *there is a strongly continuous action α of $\mathbb{T}^k$ on B such that $\alpha_z(\phi(j_X(x))) = z^{d(x)}\phi(j_X(x))$ for all $x \in X$ and $z \in \mathbb{T}^k$, and*
(2) *$\phi|_{j_X(A)} \colon A \to B$ is injective.*

Proof. Certainly $\mathbb{N}^k$ is directed, and Lemma 4.3 of [35] implies that X is $\tilde{\phi}$-injective. Since $\mathbb{Z}^k$ is amenable, $\mathcal{NO}_X$ has the gauge-invariant uniqueness property by Corollary 4.12.

Every coaction β of $\mathbb{Z}^k$ determines and is determined by a strongly continuous action α of $\mathbb{T}^k = \widehat{\mathbb{Z}^k}$: specifically, $\beta(a) = a \otimes i_{\mathbb{Z}^k}(m)$, if and only if $\alpha_z(a) = z^m a$ for all $z \in \mathbb{T}^k$. Hence condition (GI1) is equivalent, in this setting, to condition (1). □

5. Applications and examples

In this section we investigate a number of examples which illustrate both the class of C^*-algebras $\mathcal{NO}^r_X$ and the utility of its co-universal property as set out in Theorem 4.1.

Group crossed products. Let (G, P) be a quasi-lattice ordered group, and let $\alpha \colon G \to \operatorname{Aut}(A)$ be an action of G on a C^*-algebra A. Suppose that ω is a $\mathbb{T}$-valued cocycle on G; that is $\omega \colon G \times G \to \mathbb{T}$ satisfies $\omega(e, e) = 1$ and

$$\omega(gh, k)\omega(g, h) = \omega(g, hk)\omega(h, k) \text{ for all } g, h, k \in G.$$

Recall from [18, Lemma 3.2] that there is a product system $X := X(\alpha, \omega)$ over the opposite semigroup P^{op} defined as follows: for $p \in P$, let X_p be the right-Hilbert A–A bimodule which is equal to A as a normed vector space with operations

$$\langle x, y\rangle_A := x^*y \qquad a \cdot x = \alpha_p(a)x \quad \text{and} \quad x \cdot a = xa$$

for all $x, y \in X_p$ and $a \in A$, and define isomorphisms $X_p \otimes_A X_q \to X_{qp}$ by $x \otimes_A y \mapsto \overline{\omega(q, p)}\alpha_q(x)y$. Then $X(\alpha, \omega) = \coprod_{p \in P^{op}} X_p$ is the claimed product system (note that the left and right actions are compatible with the product in X because $\omega(p, e) = \omega(e, p) = 1$ for all $p \in P$). Moreover, the left action ϕ_p satisfies $\phi_p(A) \subset \mathcal{K}(X_p)$ for all $p \in P$, and since each X_p is essential in Fowler's sense, [18, Proposition 5.8] implies that $X(\alpha, \omega)$ is compactly aligned.

The twisted crossed product $A \times_{\alpha,\omega} G$ is the universal C^*-algebra generated by a covariant representation of (A, G, α, ω): that is, a homomorphism $i^{\alpha,\omega}_A \colon A \to A \times_{\alpha,\omega} G$ and multiplier unitaries $\{ i^{\alpha,\omega}_G(g) \mid g \in G \}$ such that for $g, h \in G$ and $a \in A$,

$$i^{\alpha,\omega}_G(g)i^{\alpha,\omega}_G(h) = \omega(g, h)i^{\alpha,\omega}_G(gh) \quad \text{and} \quad i^{\alpha,\omega}_G(g)i^{\alpha,\omega}_A(a)i^{\alpha,\omega}_G(g)^* = i^{\alpha,\omega}_A(\alpha_g(a));$$

we have used $(i^{\alpha,\omega}_A, i^{\alpha,\omega}_G)$ in this example, rather than the traditional (i_A, i_G) to distinguish the inclusion of G as unitaries in $A \times_{\alpha,\omega} G$ from its inclusion as unitaries in $C^*(G)$. There is a coaction $\widehat{\alpha}$ of G on $A \times_{\alpha,\omega} G$ determined by $\widehat{\alpha}(i^{\alpha,\omega}_A(a)i^{\alpha,\omega}_G(g)) = i^{\alpha,\omega}_A(a)i^{\alpha,\omega}_G(g) \otimes i_G(g)$ for all $a \in A$ and $g \in G$, and the universal properties of the two algebras involved show that the crossed product $A \times_{\alpha,\omega} G$ is isomorphic to the full cross-sectional algebra of the resulting Fell bundle. The reduced crossed product $A \times^r_{\alpha,\omega} G$ is the reduced cross-sectional algebra $(A \times_{\alpha,\omega} G)^r$ of the same bundle, and is a quotient of $A \times_{\alpha,\omega} G$. We

write $(\lambda_A^{\alpha,\omega}, \lambda_G^{\alpha,\omega})$ for the generating covariant representation of (A, G, α, ω) in $A \times^r_{\alpha,\omega} G$. Recall that $\widehat{\alpha}^n$ denotes the normalisation of $\widehat{\alpha}$ and is a normal coaction on $A \times^r_{\alpha,\omega} G$.

Lemma 5.1. *Let (G, P) be a quasi-lattice ordered group such that P is directed and G is generated as a group by P. Let $\alpha\colon G \to \mathrm{Aut}(A)$, $\omega\colon G \times G \to \mathbb{T}$ and X be as above. Then there is an isomorphism $\phi\colon A \times^r_{\alpha,\omega} G \to \mathcal{NO}^r_X$ which takes $(\lambda_G^{\alpha,\omega}(p))^*\lambda_A^{\alpha,\omega}(x)$ to $j^r_X(x)$ for all $x \in X_p = A$ and satisfies $\nu^n \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \widehat{\alpha}^n$.*

Proof. We will first show that $A \times^r_{\alpha,\omega} G$ is generated by a Nica covariant representation of X. We will then apply Theorem 4.1 to obtain a surjective homomorphism ϕ from $A \times^r_{\alpha,\omega} G$ to $\mathcal{NO}^r_X$. Finally, we will use the canonical faithful conditional expectation from $A \times^r_{\alpha,\omega} G$ to A to see that ϕ is injective.

We begin by constructing a representation ψ of X in $A \times^r_{\alpha,\omega} G$. For $p \in P$, define $\psi_p\colon X_p \to A \times^r_{\alpha,\omega} G$ by

$$\psi_p(x) := (\lambda_G^{\alpha,\omega}(p))^*\lambda_A^{\alpha,\omega}(x) \quad \text{for all } x \in X_p = A.$$

In the following calculations, we use $\diamond$ for the multiplication in P^{op}; so $p \diamond q = qp$ for all $p, q \in P$. Fix $p, q \in P$ and elements $x \in X_p$ and $y \in X_q$, and calculate:

$$\begin{aligned}
\psi(x)\psi(y) &= (\lambda_G^{\alpha,\omega}(p))^*\lambda_A^{\alpha,\omega}(x)(\lambda_G^{\alpha,\omega}(q))^*\lambda_A^{\alpha,\omega}(y)\\
&= (\lambda_G^{\alpha,\omega}(p))^*(\lambda_G^{\alpha,\omega}(q))^*(\lambda_G^{\alpha,\omega}(q))\lambda_A^{\alpha,\omega}(x)(\lambda_G^{\alpha,\omega}(q))^*\lambda_A^{\alpha,\omega}(y)\\
&= \overline{\omega(q,p)}(\lambda_G^{\alpha,\omega}(qp))^*\lambda_A^{\alpha,\omega}(\alpha_q(x))\lambda_A^{\alpha,\omega}(y)\\
&= \psi_{p\diamond q}(xy).
\end{aligned}$$

Moreover, for $p \in P$ and $x, y \in X_p$, we have

$$\begin{aligned}
\psi(x)^*\psi(y) &= \big((\lambda_G^{\alpha,\omega}(p))^*\lambda_A^{\alpha,\omega}(x)\big)^*\big((\lambda_G^{\alpha,\omega}(p))^*\lambda_A^{\alpha,\omega}(y)\big)\\
&= \lambda_A^{\alpha,\omega}(x)^*(\lambda_G^{\alpha,\omega}(p))(\lambda_G^{\alpha,\omega}(p))^*\lambda_A^{\alpha,\omega}(y) = \lambda_A^{\alpha,\omega}(x^*y) = \psi_e(\langle x, y\rangle_A).
\end{aligned}$$

Each X_p is essential, the left action of A on each X_p is injective and by compacts, and P is directed. Hence [35, Corollary 5.2] implies that ψ is CNP-covariant provided the condition $\psi^{(p)} \circ \phi_p = \lambda_A^{\alpha,\omega}$ holds for all $p \in P$. To verify this condition, fix an approximate identity $(e_k)_{k\in K}$ for A, and note that then $\phi_p(a) = \lim_{k\in K} \alpha_p(a) \otimes e_k^*$ for $p \in P$ and $a \in P$. Hence

$$\psi^{(p)}(\phi_p(a)) = \lim_{k\in K} \psi_p(\alpha_p(a))\psi_p(e_k)^* = \lim_{k\in K} \lambda_G^{\alpha,\omega}(p)^*\lambda_A^{\alpha,\omega}(\alpha_p(a)e_k^*)\lambda_G^{\alpha,\omega}(p) = \lambda_A^{\alpha,\omega}(a),$$

showing that ψ is CNP-covariant.

The image of ψ generates $A \times^r_{\alpha,\omega} G$ because the latter is spanned by elements of the form $\lambda_G^{\alpha,\omega}(g)\lambda_A^{\alpha,\omega}(a)$, and each $\lambda_G^{\alpha,\omega}(g) \in C^*(\{\, \lambda_G^{\alpha,\omega}(p) \mid p \in P \,\})$ because G is generated as a group by P. Moreover, ψ is injective as a representation because $\lambda_A^{\alpha,\omega}$ is automatically injective.

Since the left action on each fibre of X is injective, [35, Lemma 4.3] implies that X is $\tilde{\phi}$-injective. Since the normalisation $\widehat{\alpha}^n$ of $\widehat{\alpha}$ of G on $A \times^r_{\alpha,\omega} G$ satisfies $\widehat{\alpha}^n(\psi(x)) = \psi(x) \otimes i_G(p)$ for all $p \in P$ and $x \in X_p$, Theorem 4.1 gives a surjective homomorphism $\phi\colon A \times^r_{\alpha,\omega} G \to \mathcal{NO}^r_X$ such that $\phi \circ \psi = j^r_X$. Finally, the coaction $\widehat{\alpha}^n$ is normal by definition, and so Corollary 4.9 applies to give injectivity of ϕ. $\square$

Corollary 5.2. *Resume the hypotheses of Lemma 5.1. Then there is an isomorphism* $A \times_{\alpha,\omega} G \to \mathcal{NO}_X$ *which takes* $i_G^{\alpha,\omega}(p)^* i_A^{\alpha,\omega}(x)$ *to* $j_X(x)$ *for all* $x \in X_p = A$.

Proof. Since the isomorphism $\phi\colon A \times^r_{\alpha,\omega} G \to \mathcal{NO}^r_X$ of Lemma 5.1 intertwines the coactions on the two algebras, the corresponding Fell bundles are isometrically isomorphic. Since $A \times_{\alpha,\omega} G$ and $\mathcal{NO}_X$ are the full cross-sectional algebras of these Fell bundles (see Remark 4.5), the result follows. $\square$

Corollary 5.3. *Let* (G, P) *be a quasi-lattice ordered group such that* G *is generated as a group by* P. *Suppose that* P *is directed. Let* X *be the product system over* P *such that* $X_p = \mathbb{C}$ *for all* $p \in P$ *with all operations given by the usual operations in* $\mathbb{C}$. *Then* $\mathcal{NO}_X \cong C^*(G)$, *and* $\mathcal{NO}^r_X \cong C^*_r(G)$. *Specifically, if* 1_p *denotes the element* $1 \in \mathbb{C}$ *when regarded as an element of* X_p, *then* $j_X(1_p) \mapsto i_G(p)$ *determines an isomorphism* $\mathcal{NO}_X \cong C^*(G)$, *and* $j^r_X(1_p) \mapsto \lambda_G(p)$ *determines an isomorphism* $\mathcal{NO}^r_X \cong C^*_r(G)$.

Proof. We apply Lemma 5.1 and Corollary 5.2 to the trivial action α of P on $\mathbb{C}$ and the trivial cocycle ω on G. $\square$

Remark 5.4. Let G be a nonabelian finite-type Artin group and P its standard positive cone. By [5, Proposition 29], P is directed and G is not amenable. Therefore Corollary 5.3 implies that for the product system considered there, we have

$$\mathcal{NO}_X \cong C^*(G) \ncong C^*_r(G) \cong \mathcal{NO}^r_X$$

(cf. [5, Theorem 30]). In particular $\mathcal{NO}_X$ does not have the gauge-invariant uniqueness property; so the gauge-invariant uniqueness property is not automatic even for systems where the left action is compact and injective and P is directed. We thank Marcelo Laca for bringing this example to our attention.

Boundary quotient algebras. The results in this section refer to the boundary quotient algebras studied in [6]. Throughout, given a quasi-lattice ordered group (G, P), we write Ω for the Nica spectrum of (G, P) and α for the partial action of G on Ω considered in [6] (see also [15]), and we let $\partial\Omega$ be the boundary of Ω defined in [6] and [24].

If α is a partial action of a discrete group G on a C^*-algebra A, [31, Proposition 3.2] shows that there is a canonical (dual) coaction $\widehat{\alpha}$ on the full partial crossed product $A \rtimes_\alpha G$. Moreover, the discussion following [31, Remark 3.7] shows that the normalisation of $\widehat{\alpha}$ is naturally a coaction on the reduced partial crossed product $A \rtimes_r G$.

Lemma 5.5. *Let* (G, P) *be a quasi-lattice ordered group. Let* $C_0(\partial\Omega) \rtimes_r G$ *be the reduced partial crossed product corresponding to the partial crossed product* $C_0(\partial\Omega) \rtimes G$ *considered in* [6], *and let* $\beta\colon C_0(\partial\Omega) \rtimes_r G \to (C_0(\partial\Omega) \rtimes_r G) \otimes C^*(G)$ *be the canonical coaction of* G *on* $C_0(\partial\Omega) \rtimes_r G$. *Let* X *be the product system over* P *such that* $X_p = \mathbb{C}$ *for all* p. *Then there is an isomorphism* $\phi\colon C_0(\partial\Omega) \rtimes_r G \to \mathcal{NO}^r_X$ *such that* $\nu^n \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \beta$. *In particular,* $\phi|_{C_0(\partial\Omega)}$ *is an isomorphism from* $C_0(\partial\Omega)$ *to* $(\mathcal{NO}^r_X)^{\nu^n}_e$.

Proof. Let (π, u) be the universal covariant representation of $(C(\Omega), G, \alpha)$. By [15, Proposition 6.1 and Theorem 6.4] the collection $\{\, u(p) \mid p \in P \,\}$ is a family of isometries satisfying Nica's covariance relation which generates $C(\Omega) \rtimes_\alpha G$. Since $C_0(\partial\Omega) \rtimes_r G$ is a quotient of $C(\Omega) \rtimes_\alpha G$, it follows that $C_0(\partial\Omega) \rtimes_r G$ is generated by nonzero isometries $\{\, W_p \mid p \in P \,\}$ (that these isometries are nonzero follows from the fact that $\partial\Omega$

is nonempty, cf. [6, Lemma 3.5] and [24, Theorem 3.7]) satisfying Nica's covariance relation such that the canonical coaction β satisfies $\beta(W_p) = W_p \otimes \lambda_G(p)$ for all p. For $p \in P$ let 1_p denote the complex number 1 regarded as an element of X_p. The assignment $\psi \mapsto \{ \psi(1_p) \mid p \in P \}$ is a bijective correspondence between injective Nica covariant Toeplitz representations of X, and families of nonzero isometries satisfying Nica's covariance relation. Thus $C_0(\partial\Omega) \rtimes_r G$ is generated by an injective Nica covariant Toeplitz representation ψ of X satisfying $\beta(\psi(x)) = \psi(x) \otimes i_G(d(x))$ for all $x \in X$. Since the left action on each fibre of X is implemented by an injective homomorphism, Theorem 4.1 gives a surjective homomorphism $\phi\colon C_0(\partial\Omega) \rtimes_r G \to \mathcal{NO}_X^r$ such that $\nu^n \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \beta$, and we need only show that ϕ is injective.

By [6, Lemma 3.5], $\partial\Omega$ is the unique minimal closed invariant subset of the Nica spectrum Ω, and since ϕ is nonzero, it follows that ϕ is injective on $C_0(\partial\Omega)$. Since β is normal, the expectation $\Phi^\beta\colon C_0(\partial\Omega) \rtimes_r G \to C_0(\partial\Omega)$ is faithful. Since $\nu^n \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \beta$, it follows that ϕ intertwines the expectation Φ^β and the expectation from $\mathcal{NO}_X^r$ to $(\mathcal{NO}_X^r)_e^{\nu^n}$. The standard argument now shows that ϕ is injective. □

Corollary 5.6. *Resume the hypotheses of Lemma 5.5. Let $\overline{\beta}$ be the canonical coaction on the universal partial crossed product C^*-algebra $C_0(\partial\Omega) \rtimes G$. There is an isomorphism $\phi\colon C_0(\partial\Omega) \rtimes G \to \mathcal{NO}_X$ such that $\nu \circ \phi = (\phi \otimes \mathrm{id}_{C^*(G)}) \circ \overline{\beta}$.*

Proof. We use the same trick as in the proof of Corollary 5.2. □

Remark 5.7. The proofs of Lemma 5.5 and Corollary 5.6 are excellent examples of the utility of the co-universal property of $\mathcal{NO}_X^r$. To prove the same results otherwise one would first have to show that $\mathcal{NO}_X$ is isomorphic to the universal partial crossed product of $C_0(\partial\Omega)$ by G and then argue that the normalisations of the two coactions on these universal C^*-algebras yield $\mathcal{NO}_X^r$ and $C_0(\partial\Omega) \rtimes_r G$. Moreover, proving equality of universal C^*-algebras would require verifying condition (CP) of [35] in one direction, and the elementary relations associated with the essential spectrum from [6] in the other direction — for non-amenable G, there would be no gauge-invariant uniqueness theorem to apply in either direction.

Remark 5.8. Combining Lemma 5.5 with Lemma 5.1, we see that if (G, P) is a quasi-lattice ordered group such that G is generated as a group by P and each pair of elements in P has a common upper bound, then the boundary quotient algebra $C_0(\partial\Omega) \rtimes_r G$ is isomorphic to the reduced group C^*-algebra $C_r^*(G)$; and under the same hypotheses, Corollary 5.6 combined with Corollary 5.2 shows that the universal partial crossed product C^*-algebra $C_0(\partial\Omega) \rtimes G$ is isomorphic to the full group C^*-algebra $C^*(G)$.

Topological higher-rank graphs. In this section we show that each compactly aligned topological higher-rank graph Λ in the sense of Yeend can be used to construct a compactly aligned product system X of Hilbert bimodules over $C_0(\Lambda^0)$ with resulting $\mathcal{T}_{\mathrm{cov}}(X)$ isomorphic to the C^*-algebra of Yeend's path groupoid and with corresponding $\mathcal{NO}_X$ isomorphic to the C^*-algebra of Yeend's boundary-path groupoid.

Recall [37] that, for $k \in \mathbb{N}$, a *topological k-graph* is a pair (Λ, d) consisting of: (1) a small category Λ endowed with a second countable locally compact Hausdorff topology under which the composition map is continuous and open, the range map r is continuous and the source map s is a local homeomorphism; and (2) a continuous functor $d\colon \Lambda \to$

$\mathbb{N}^k$, called the *degree map*, satisfying the factorisation property: if $d(\lambda) = m + n$ then there exist unique μ, ν with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Elements of Λ are called paths, and paths of degree 0 are called vertices. For $m \in \mathbb{N}^k$ we define $\Lambda^m := d^{-1}(m)$. If $0 \le m \le n \le p$ in $\mathbb{N}^k$ and $\lambda \in \Lambda^p$ then we write $\lambda(m, n)$ for the unique path in Λ^{n-m} such that $\lambda = \mu\lambda(m, n)\nu$, where $\mu \in \Lambda^m$ and $\nu \in \Lambda^{p-n}$. For $0 \le m \le p$ in $\mathbb{N}^k$ and $\lambda \in \Lambda^p$ we write $\lambda(m)$ for $s(\lambda(0, m)) = \lambda(m, m)$. If $m, p \in \mathbb{N}^k$ with $m \le p$ then the map $\sigma^m \colon \Lambda^p \to \Lambda^{p-m}$ such that $\sigma^m(\lambda) = \lambda(m, p)$ is continuous. For $U, V \subset \Lambda$, we write

$$UV := \{ \lambda\mu \mid \lambda \in U,\ \mu \in V,\ s(\lambda) = r(\mu) \}.$$

For $U \subset \Lambda^p$ and $V \subset \Lambda^q$,

$$U \vee V := U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q}$$

is the set of *minimal common extensions* of paths from U and V. A topological k-graph (Λ, d) is *compactly aligned* if $U \vee V$ is compact whenever U and V are compact.

Fix $k \in \mathbb{N}$, and let (Λ, d) be a topological k-graph. Define $A := C_0(\Lambda^0)$. For each $n \in \mathbb{N}^k$ let X_n be the right-Hilbert A-A bimodule associated to the topological graph $(\Lambda^0, \Lambda^n, s|_{\Lambda^n}, r|_{\Lambda^n})$ (see [21]). So X_n is the completion of the pre-Hilbert A–A bimodule $C_c(\Lambda^n)$ with operations

$$\langle f, g\rangle_A^n(v) = \textstyle\sum_{\eta\in\Lambda^n v}\overline{f(\eta)}g(\eta) \quad \text{and} \quad (a\cdot f\cdot b)(\lambda) = a(r(\lambda))f(\lambda)b(s(\lambda)).$$

Katsura shows that X_n is a subspace of $C_0(\Lambda^n)$ [21, Section 1].

Proposition 5.9. *Let Λ be a topological k-graph and let $(X_n)_{n\in\mathbb{N}^k}$ be as above. For $f \in X_m$ and $g \in X_n$, define $fg \colon \Lambda^{m+n} \to \mathbb{C}$ by $(fg)(\lambda) := f(\lambda(0, m))g(\lambda(m, m + n))$. Under this multiplication the family $X := \bigsqcup_{n\in\mathbb{N}^k} X_n$ of right-Hilbert $C_0(\Lambda^0)$-$C_0(\Lambda^0)$ bimodules is a product system over $\mathbb{N}^k$.*

Proof. We first show that for $f_1, f_2 \in X_m$ and $g_1, g_2 \in X_n$ we have $\langle f_1g_1, f_2g_2\rangle_A^{m+n} = \langle g_1, \langle f_1, f_2\rangle_A^m \cdot g_2\rangle_A^n$. The functions f_1g_1, f_2g_2 are continuous on Λ^{m+n} and for $v \in \Lambda^0$,

$$\begin{aligned}\langle f_1g_1, f_2g_2\rangle_A^{m+n}(v) &= \textstyle\sum_{\lambda\in\Lambda^{m+n}v}\overline{f_1(\lambda(0,m))}f_2(\lambda(0,m))\overline{g_1(\lambda(m,m+n))}g_2(\lambda(m,m+n))\\ &= \textstyle\sum_{\nu\in\Lambda^n v}\big(\sum_{\mu\in\Lambda^m r(\nu)}\overline{f_1(\mu)}f_2(\mu)\big)\overline{g_1(\nu)}g_2(\nu)\\ &= \textstyle\sum_{\nu\in\Lambda^n v}\langle f_1, f_2\rangle_A^m(r(\nu))\overline{g_1(\nu)}g_2(\nu)\\ &= \langle g_1, \langle f_1, f_2\rangle_A^m \cdot g_2\rangle_A^n(v).\end{aligned}$$

Taking $f_2 = f_1 = f$ and $g_2 = g_1 = g$, we deduce that $fg \in X_{m+n}$ by definition of X_{m+n} (see [21]). Further, since $\langle g_1, \langle f_1, f_2\rangle_A^m \cdot g_2\rangle_A^n = \langle f_1 \otimes_A g_1, f_2 \otimes_A g_2\rangle_A$, the map $f \otimes_A g \mapsto fg$ extends to an isometric adjointable operator from $X_m \otimes_A X_n$ to X_{m+n}. To show that it is surjective it is enough to show that it has dense range. The subset $\mathcal{A} := \operatorname{span}\{ fg \mid f \in C_c(\Lambda^m), g \in C_c(\Lambda^n) \}$ of $C_c(\Lambda^{m+n})$ is a subalgebra of $C_0(\Lambda^{m+n})$. For every open subset U of Λ^{m+n}, an application of the Stone-Weierstrass Theorem shows that $\mathcal{A} \cap C_c(U)$ is uniformly dense in $C_c(U)$. So $\mathcal{A}$ is dense in X_{m+n} with respect to $\|\cdot\|_A$ by [21, Lemma 1.26], and the multiplication operator has dense range. $\square$

To apply our results, we need to show that the product system X is compactly aligned if Λ is compactly aligned. We first need a couple of technical lemmas.

Notation 5.10. For $m \in \mathbb{N}^k$ we denote by F_m the set of functions $f \in C_c(\Lambda^m)$ such that the source map restricts to a homeomorphism of $\operatorname{supp}(f)$. By definition of F_m, for each $f \in F_m$ and $v \in \Lambda^0$ such that $\Lambda^m v$ is non-empty we may fix an element, henceforth denoted $\lambda_{f,v}$, of $\Lambda^m v$ such that $f(\mu) = 0$ for all $\mu \in \Lambda^m v \setminus \{\lambda_{f,v}\}$.

Lemma 5.11. *For $m \in \mathbb{N}^k$, $\operatorname{span} F_m$ is dense in X_m with respect to the norm $\|\cdot\|_A$.*

Proof. A partition of unity argument using that the source map in Λ is a local homeomorphism shows that each element of $C_c(\Lambda^m)$ is a finite sum of elements of F_m, and the result follows. $\square$

Lemma 5.12. *Fix $m, n \in \mathbb{N}^k$. Then, for $f \in X_m$, $g \in F_m$, $c \in X_{m\vee n}$, and $\xi \in \Lambda^{m\vee n}$ we have*

$$\big(\iota_m^{m\vee n}(f \otimes g^*)(c)\big)(\xi) = f(\xi(0,m))\overline{g(\lambda_{g,\xi(m)})}c(\lambda_{g,\xi(m)}\xi(m, m\vee n)).$$

Proof. For $h \in X_m$ and $l \in X_{(m\vee n)-m}$ we have

$$\begin{aligned}\big(\iota_m^{m\vee n}(f \otimes g^*)(hl)\big)(\xi) &= f(\xi(0,m))\big(\textstyle\sum_{\mu\in\Lambda^m\xi(m)}\overline{g(\mu)}h(\mu)\big)l(\xi(m, m\vee n))\\ &= f(\xi(0,m))\overline{g(\lambda_{g,\xi(m)})}h(\lambda_{g,\xi(m)})l(\xi(m, m\vee n))\\ &= f(\xi(0,m))\overline{g(\lambda_{g,\xi(m)})}hl(\lambda_{g,\xi(m)}\xi(m, m\vee n)).\end{aligned}$$

Since elements of the form hl are dense in $X_{m\vee n}$, the result follows. $\square$

Lemma 5.13 (cf. [32, Lemma 5.1])**.** *Fix $m \in \mathbb{N}^k$ and $T \in \mathcal{K}(X_m)$. Let $B_1F_m := \{f \in F_m \mid ||f||_\infty \leq 1\}$. Then the function $\chi_T : \Lambda^m \to \mathbb{R}$ defined by $\chi_T(\lambda) = \sup_{f\in B_1F_m}|T(f)(\lambda)|$ vanishes at infinity on X_m.*

Proof. Since $\chi_{\alpha S+\beta T}(\lambda) \leq |\alpha|\chi_S(\lambda)+|\beta|\chi_T(\lambda)$, and $|T(f)(\lambda)| \leq \|T\|\,\|f\|_{X_m}$ for $\alpha, \beta \in \mathbb{C}$, $S, T \in \mathcal{K}(X_m)$, $f \in F_m$ and $\lambda \in \Lambda^m$, it is enough to prove that χ_T vanishes at infinity when $T = g \otimes h^*$ for $g, h \in X_m$.

If $g, h \in X_m$, $f \in B_1F_m$ and $\lambda \in \Lambda^m$, then we have

$$|(g \otimes h^*)(f)(\lambda)| = \big|g(\lambda)\textstyle\sum_{\eta\in\Lambda^m s(\lambda)}\overline{h(\eta)}f(\eta)\big| = |g(\lambda)\overline{h(\lambda_{f,s(\lambda)})}f(\lambda_{f,s(\lambda)})| \leq |g(\lambda)|\|h\|_\infty$$

and since g vanishes at infinity on Λ^m, so will $\chi_{g\otimes h^*}$. $\square$

Most of the work in proving that X is compactly aligned when Λ when is compactly aligned, is involved in proving the following technical lemma. We state this lemma separately because we will use it again to prove Proposition 5.19.

Lemma 5.14. *Assume that Λ is compactly aligned. Fix $m, n \in \mathbb{N}^k$. Let $f_m \in F_m$ and $f_n \in F_n$. Then $C := \operatorname{supp}(f_m) \vee \operatorname{supp}(f_n) \subset \Lambda^{m\vee n}$ is compact. For each of $p = m, n$: let $\{\, V_i^p \mid 1 \leq i \leq r_p \,\}$ be a finite open cover of $\sigma^p(C)$ such that each $\overline{V_i^p}$ is compact and s restricts to a homeomorphism on each V_i^p; let $\phi_i^p \colon \sigma^p(C) \to [0,1]$, $1 \leq i \leq r_p$, be a partition of unity on $C(C)$ subordinate to $\{\, V_i^p \cap \sigma^p(C) \mid 1 \leq i \leq r_p \,\}$; and fix functions $\rho_i^p \colon \Lambda^{(m\vee n)-p} \to [0,1]$ such that each $\rho_i^p|_{\sigma^p(C)} = \sqrt{\phi_i^p}$, and each ρ_i^p vanishes off V_i^p. Fix $g_m \in F_m$ and $g_n \in F_n$. For $1 \leq i \leq r_m$ and $1 \leq j \leq r_n$, define $a_{ij}, b_j \in C_c(\Lambda^{m\vee n})$ by $a_{ij} := g_m(\rho_i^m \cdot \langle f_m\rho_i^m, f_n\rho_j^n\rangle_A^{m\vee n})$ and $b_j := g_n\rho_j^n$. Then*

$$\iota_m^{m\vee n}(g_m \otimes f_m^*)\iota_n^{m\vee n}(f_n \otimes g_n^*) = \textstyle\sum_{i=1}^{r_m}\sum_{j=1}^{r_n}a_{ij} \otimes b_j^*. \tag{5.1}$$

Proof. Since Λ is compactly aligned, C is compact. Since the source map is injective on each $\operatorname{supp}(\rho_i^m)$, for $\mu \in \Lambda^{(m\vee n)-m}$ such that $\Lambda^m r(\mu)$ is non-empty and $1 \le j \le r_n$ we have

$$\begin{aligned}&\big(\textstyle\sum_{i=1}^{r_m}\rho_i^m \cdot \langle f_m\rho_i^m, f_n\rho_j^n\rangle_A^{m\vee n}\big)(\mu)\\ &\quad= \textstyle\sum_{i=1}^{r_m}\rho_i^m(\mu)\sum_{\alpha\in\Lambda^{m\vee n}s(\mu)}\overline{f_m(\alpha(0,m))}\rho_i^m(\alpha(m,m\vee n))(f_n\rho_j^n)(\alpha)\\ &\quad= \textstyle\sum_{i=1}^{r_m}(\rho_i^m(\mu))^2\overline{f_m(\lambda_{f_m,r(\mu)})}(f_n\rho_j^n)(\lambda_{f_m,r(\mu)}\mu)\\ &\quad= \overline{f_m(\lambda_{f_m,r(\mu)})}(f_n\rho_j^n)(\lambda_{f_m,r(\mu)}\mu).\end{aligned}$$

Fix $c \in X_{m\vee n}$ and $\xi \in \Lambda^{m\vee n}$, and write $\lambda_m := \lambda_{f_m,\xi(m)}$, $\lambda' := \xi(m, m\vee n)$, and $\beta := (\lambda_m\lambda')(n, m\vee n)$. If $\beta \in \sigma^n(C)$ then, since the source map is injective on each $\operatorname{supp}(\rho_j^n)$, we have

$$\begin{aligned}\textstyle\sum_{j=1}^{r_n}\rho_j^n(\beta)\langle b_j, c\rangle_A^{m\vee n}(s(\xi)) &= \textstyle\sum_{j=1}^{r_n}\rho_j^n(\beta)\sum_{\lambda\in\Lambda^{m\vee n}s(\xi)}\overline{g_n(\lambda(0,n))}\rho_j^n(\lambda(n,m\vee n))c(\lambda)\\ &= \textstyle\sum_{j=1}^{r_n}(\rho_j^n(\beta))^2\overline{g_n(\lambda_{g_n,r(\beta)})}c(\lambda_{g_n,r(\beta)}\beta)\\ &= \overline{g_n(\lambda_{g_n,r(\beta)})}c(\lambda_{g_n,r(\beta)}\beta).\end{aligned}$$

Hence we have

$$\begin{aligned}&\big(\textstyle\sum_{i=1}^{r_m}\sum_{j=1}^{r_n} a_{ij}\otimes b_j^*\big)(c)(\xi)\\ &\quad= \textstyle\sum_{i=1}^{r_m}\sum_{j=1}^{r_n} g_m(\xi(0,m))(\rho_i^m\cdot\langle f_m\rho_i^m, f_n\rho_j^n\rangle_A^{m\vee n})(\lambda')\langle b_j, c\rangle_A^{m\vee n}(s(\xi))\\ &\quad= g_m(\xi(0,m))\textstyle\sum_{j=1}^{r_n}(\sum_{i=1}^{r_m}\rho_i^m\cdot\langle f_m\rho_i^m, f_n\rho_j^n\rangle_A^{m\vee n})(\lambda')\langle b_j, c\rangle_A^{m\vee n}(s(\xi))\\ &\quad= g_m(\xi(0,m))\overline{f_m(\lambda_m)}f_n((\lambda_m\lambda')(0,n))\overline{g_n(\lambda_{g_n,r(\beta)})}c(\lambda_{g_n,r(\beta)}\beta)\end{aligned}$$

which equals $\big(\iota_m^{m\vee n}(g_m\otimes f_m^*)\iota_n^{m\vee n}(f_n\otimes g_n^*)\big)(c)(\xi)$ by two applications of Lemma 5.12. $\square$

The next result generalises [32, Theorem 5.4].

Proposition 5.15. *Let Λ be a topological k-graph. The product system X defined by Proposition 5.9 is compactly aligned if and only if Λ is compactly aligned.*

Proof. Assume that Λ is compactly aligned. Fix $m, n \in \mathbb{N}^k$. By Lemma 5.11 it suffices to show that, for $f, g \in F_m$ and $h, l \in F_n$, $\iota_m^{m\vee n}(f\otimes g^*)\iota_n^{m\vee n}(h\otimes l^*) \in \mathcal{K}(X_{m\vee n})$, and this follows from Lemma 5.14.

Assume next that Λ is not compactly aligned. Then there exist $m, n \in \mathbb{N}^k$ and $U \subset \Lambda^m$, $V \subset \Lambda^n$ such that U and V are compact, but $U \vee V$ is not compact. For each of $C = U, V$: let $\{\, V_i^C \mid 1 \le i \le r_C \,\}$ be a finite open cover of C such that each $\overline{V_i^C}$ is compact and s restricts to a homeomorphism on each V_i^C; let $\phi_i^C\colon C \to [0,1]$, $1 \le i \le r_C$, be a partition of unity on $C(C)$ subordinate to $\{\, V_i^C \cap C \mid 1 \le i \le r_C \,\}$; and fix functions $\rho_i^C \in C_C(\Lambda^p)$, where $p = m$ if $C = U$ and $p = n$ if $C = V$, such that each $\rho_i^C|_C = \sqrt{\phi_i^C}$, and each ρ_i^C vanishes off V_i^C. Let $T_C = \sum_{i=1}^{r_C}\rho_i^C \otimes (\rho_i^C)^*$. Then $T_U \in \mathcal{K}(X_m)$ and $T_V \in \mathcal{K}(X_n)$, but we claim that $T := \iota_m^{m\vee n}(T_U)\iota_n^{m\vee n}(T_V)$ is not compact which will show that X is not compactly aligned. Notice first that if $f \in X_p$

and $\lambda \in C$, then we have

$$
\begin{aligned}
T_C(f)(\lambda) &= \sum_{i=1}^{r_C} \big(\rho_i^C \otimes (\rho_i^C)^*\big)(f)(\lambda) \\
&= \sum_{i=1}^{r_C} \rho_i^C(\lambda) \sum_{\eta \in \Lambda^p s(\lambda)} \overline{\rho_i^C(\eta)} f(\eta) = \sum_{i=1}^{r_C} \rho_i^C(\lambda)\overline{\rho_i^C(\lambda)} f(\lambda) = f(\lambda).
\end{aligned}
\tag{5.2}
$$

For each $\lambda \in U \vee V$ choose $f_\lambda \in B_1 F_{m\vee n}$ such that $f_\lambda(\lambda) = 1$. Equation (5.2) implies that

$$T(f_\lambda)(\lambda) = \iota_m^{m\vee n}(T_U)\iota_n^{m\vee n}(T_V)(f_\lambda)(\lambda) = \iota_n^{m\vee n}(T_V)(f_\lambda)(\lambda) = f_\lambda(\lambda) = 1,$$

so it follows from Lemma 5.13 that T is not compact. □

In [37], Yeend associated two groupoids G_Λ and $\mathcal{G}_\Lambda$ and hence two C^*-algebras $C^*(G_\Lambda)$ and $C^*(\mathcal{G}_\Lambda)$ to each compactly aligned topological higher-rank graph Λ, and proposed $C^*(G_\Lambda)$ as a model for the Toeplitz algebra of Λ, and $C^*(\mathcal{G}_\Lambda)$ as the Cuntz-Krieger algebra. We will show that $\mathcal{T}_{\mathrm{cov}}(X)$ is isomorphic to $C^*(G_\Lambda)$ and that $\mathcal{NO}_X$ is isomorphic to $C^*(\mathcal{G}_\Lambda)$.

In the following, we use the notation for paths in topological k-graphs established in [37, Lemma 3.3]. We denote by G_Λ the *path groupoid* associated to Λ [37, Definition 3.4]. So G_Λ consists of triples (x, m, y) where x and y are (possibly infinite) paths in Λ, $m \in \mathbb{Z}^k$, and there exist $p, q \in \mathbb{N}^k$ such that $p \le d(x)$, $q \le d(y)$, $p - q = m$ and $\sigma^p(x) = \sigma^q(y)$. By [37, Theorem 3.16], G_Λ is a locally compact r-discrete topological groupoid admitting a Haar system consisting of counting measures. A basis for the topology on G_Λ is as follows [37, Proposition 3.6]. Define $\Lambda *_s \Lambda := \{\, (\lambda, \mu) \in \Lambda \times \Lambda \mid s(\lambda) = s(\mu) \,\}$, and for $U, V \subset \Lambda$ define $U *_s V := (U \times V) \cap (\Lambda *_s \Lambda)$. For $F \subset \Lambda *_s \Lambda$ and $m \in \mathbb{Z}^k$, define $Z(F, m) := \{\, (\lambda x, d(\lambda) - d(\mu), \mu x) \in G_\Lambda \mid (\lambda, \mu) \in F, d(\lambda) - d(\mu) = m \,\}$. Then the family of sets of the form $Z(U *_s V, m) \cap Z(F, m)^c$, where $m \in \mathbb{Z}^k$, $U, V \subset \Lambda$ are open and $F \subset \Lambda *_s \Lambda$ is compact, is a basis for the topology on G_Λ.

Let $V \subset \Lambda^0$. A set $E \subset V\Lambda$ is called *exhaustive* (cf. [37, Definition 4.1]) for V if for all $\lambda \in V\Lambda$ there exists $\mu \in E$ such that $\{\lambda\} \vee \{\mu\} \neq \emptyset$. For $v \in \Lambda^0$, let $v\mathcal{CE}(\Lambda)$ denote the set of all compact sets $E \subset \Lambda$ such that $r(E)$ is a neighbourhood of v and E is exhaustive for $r(E)$. A (possibly infinite) path x is called a *boundary path* (cf. [37, Definition 4.2]) if for all $m \in \mathbb{N}^k$ with $m \le d(x)$, and for all $E \in x(m)\mathcal{CE}(\Lambda)$, there exists $\lambda \in E$ such that $x(m, m + d(\lambda)) = \lambda$. We write $\partial\Lambda$ for the set of all boundary paths. It is shown in [37] that $\partial\Lambda$ is a closed and invariant subset of $G_\Lambda^{(0)}$ (we are here identifying a path x with the element $(x, 0, x)$ in $G_\Lambda^{(0)}$) and that $v\partial\Lambda \neq \emptyset$ for all $v \in \Lambda^0$. The *boundary-path groupoid* $\mathcal{G}_\Lambda$ is then defined in [37, Definition 4.8] to be the reduction of G_Λ to $\partial\Lambda$. We will now show that $\partial\Lambda$ is in fact the smallest closed and invariant subset Y of $G_\Lambda^{(0)}$ such that vY is nonempty for all $v \in \Lambda^0$. Let X_Λ denote the collection of finite and infinite paths in Λ. For $V \subset \Lambda$ write $Z(V) := \{x \in X_\Lambda \mid \text{there exists } \lambda \in V \text{ such that } x(0, d(\lambda)) = \lambda\}$.

Proposition 5.16. *Let Λ be a compactly aligned topological k-graph. Then $\partial\Lambda$ is the smallest closed and invariant subset Y of $G_\Lambda^{(0)}$ such that vY is nonempty for all $v \in \Lambda^0$.*

Proof. It follows from [37, Propositions 4.3, 4.4 and 4.7] that $\partial\Lambda$ is a closed and invariant subset of $G_\Lambda^{(0)}$ and that $v\partial\Lambda \neq \emptyset$ for all $v \in \Lambda^0$. We will show that $\partial\Lambda$ is contained in

any other closed and invariant subset Y of $G_\Lambda^{(0)}$ satisfying that vY is nonempty for all $v \in \Lambda^0$. So let Y be such a subset and assume for contradiction that there is a boundary path x such that $x \notin Y$. Since Y is closed, it follows from [37, Lemma 3.8] that there is a relatively compact and open subset U of Λ and a compact subset F of Λ satisfying that $\mu \in F$ implies that $\mu = \lambda\mu'$ for some $\lambda \in \overline{U}$ such that $x \in Z(U) \setminus Z(F) \subset X_\Lambda \setminus Y$. We will show that there is a $\lambda \in \Lambda$ such that $\lambda X_\Lambda \subset Z(U) \setminus Z(F) \subset X_\Lambda \setminus Y$. It will then follow from the invariance of Y that $s(\lambda)Y = \emptyset$, and we have our contradiction.

Choose $m \in \mathbb{N}^k$ such that $x(0,m) \in U$ and a compact neighbourhood $V \subset U \cap \Lambda^m$ of $x(0,m)$ such that s restricted to V is injective. We let

$$C := \sigma^m((V \vee F) \cap F) = \{\mu \in \Lambda \mid \text{there exists } \lambda \in V \text{ such that } \lambda\mu \in F\}.$$

Since Λ is compactly aligned and σ^m is continuous, the set C is compact. It follows from the assumption $x \in \partial\Lambda$ that if $C \in x(m)\mathcal{CE}(\Lambda)$, then there exists $\mu \in C$ such that $x(m, m+d(\mu)) \in C$, and then $x(0, m+d(\mu)) = x(0,m)\mu \in F$, a contradiction. Thus either $r(C)$ is not a neighbourhood of $x(m)$ or C is not exhaustive for $r(C)$.

If $r(C)$ is not a neighbourhood of $x(m)$, then since $s(V)$ is a neighbourhood of $x(m)$ there exists $\lambda \in V$ such that $s(\lambda) \neq r(\mu)$ for all $\mu \in C$, and then $\lambda X_\Lambda \subset Z(V) \setminus Z(F) \subset Z(U) \setminus Z(F)$.

If C is not exhaustive for $r(C)$, then there exists $\mu \in r(C)\Lambda$ such that $\mu\Lambda \cap C\Lambda = \emptyset$. Let $\lambda := \eta\mu$ where η is the unique element of V such that $s(\eta) = r(\mu)$. Then $\lambda X_\Lambda \subset Z(V) \setminus Z(F) \subset Z(U) \setminus Z(F)$. □

Proposition 5.17. *Let Λ be a compactly aligned topological k-graph. Let X be the product system constructed in Proposition 5.9. There is a unique Toeplitz representation $\psi\colon X \to C^*(G_\Lambda)$ such that, for $n \in \mathbb{N}^k$ and $f \in C_c(\Lambda^n)$, we have $\psi(f) \in C_c(G_\Lambda)$ and*

$$\psi(f)((x,p,y)) = \begin{cases} f(x(0,n)) & \text{if } p = n \text{ and } \sigma^n(x) = y \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Moreover, the $\psi(f)$ for $n \in \mathbb{N}^k$ and $f \in C_c(\Lambda^n)$ generate $C^*(G_\Lambda)$.

Proof. We first show that, for $f \in C_c(\Lambda^n)$, $\psi(f)$ defined as in (5.3) is in $C_c(G_\Lambda)$. To see that $\psi(f)$ is continuous, fix $(x,p,y) \in G_\Lambda$ and $\epsilon > 0$. If $p \neq n$ then $Z(\Lambda *_s \Lambda, p)$ is an open neigbourhood of (x,p,y) on which $\psi(f)$ is zero. If $p = n$ and $\sigma^n(x) \neq y$ then, since $U *_s s(U)$ is compact where $U := \operatorname{supp}(f)$, the subset $Z(\Lambda *_s \Lambda, n) \cap Z(U *_s s(U), n)^c$ is an open neighbourhood of (x,p,y) on which $\psi(f)$ is zero. Suppose now that $p = n$ and $\sigma^n(x) = y$. Since f is continuous there exists an open neighbourhood $U \subset \Lambda^n$ of $x(0,n)$ such that $\lambda \in U$ implies $|f(\lambda) - f(x(0,n))| < \epsilon$. Note that, since Λ^n is open in Λ, U is open in Λ. Choose an open neighbourhood V of $x(0,n)$ in Λ such that $V \subset U$, $s(V)$ is open in Λ, and $s|_V$ is a homeomorphism onto $s(V)$. Then $Z(V *_s s(V), n)$ is an open neighbourhood of (x,p,y) such that $(w,n,z) \in Z(V *_s s(V), n)$ implies $\sigma^n(w) = z$ and $w(0,n) \in V \subset U$, hence $|\psi(f)((w,n,z)) - \psi(f)((x,p,y))| = |f(w(0,n)) - f(x(0,n))| < \epsilon$. It follows that $\psi(f) \in C(G_\Lambda)$.

To see that $\psi(f)$ has compact support, let $U := \operatorname{supp}(f)$, then $Z(U *_s s(U), n)$ is compact by [37, Proposition 3.15] and $\operatorname{supp}(\psi(f)) \subset Z(U *_s s(U), n)$.

We now show that $\|\psi(f)\|_{C^*(G_\Lambda)} \leq \|f\|_A$ for $f \in C_c(\Lambda^n)$. By [34, II.1.5] it suffices to show that

$$\sup_{(y,0,y)\in G_\Lambda^{(0)}} \int_{G_\Lambda} |\psi(f)^* * \psi(f)((x,p,z))| \, d\lambda^{(y,0,y)}((x,p,z)) \leq \|f\|_A^2 \quad \text{and} \tag{5.4}$$

$$\sup_{(y,0,y)\in G_\Lambda^{(0)}} \int_{G_\Lambda} |\psi(f)^* * \psi(f)((x,p,z)^{-1})| \, d\lambda^{(y,0,y)}((x,p,z)) \leq \|f\|_A^2. \tag{5.5}$$

Fix $(y,p,z) \in G_\Lambda$. Since the Haar system on G_Λ consists of counting measures,

$$\begin{aligned}\psi(f)^* * \psi(f)((y,p,z)) &= \textstyle\sum_{(y,r,w)\in G_\Lambda} \overline{\psi(f)((w,-r,y))}\psi(f)((w,p-r,z)) \\ &= \textstyle\sum_{\alpha\in\Lambda^n r(y)} \overline{f(\alpha)}\psi(f)((\alpha y, p+n, z)) \\ &= \begin{cases} \langle f,f\rangle_A^n(r(y)) & \text{if } p=0 \text{ and } z=y \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

So, for $(y,0,y) \in G_\Lambda^{(0)}$, we have

$$\begin{aligned}&\int_{G_\Lambda} |\psi(f)^* * \psi(f)((x,p,z))| \, d\lambda^{(y,0,y)}((x,p,z)) \\ &\qquad = |\psi(f)^* * \psi(f)((y,0,y))| = \langle f,f\rangle_A^n(r(y)) \leq \|f\|_A^2,\end{aligned}$$

and this establishes (5.4). A similar argument establishes (5.5).

Straightforward calculations show that $\psi\colon C_c(\Lambda^n) \to C^*(G_\Lambda)$ is linear and is multiplicative when $n=0$. Since $\|\psi(f)\|_{C^*(G_\Lambda)} \leq \|f\|_A$ for $f \in C_c(\Lambda^n)$, ψ extends to a linear map ψ from X_n to $C^*(G_\Lambda)$, and $\psi\colon A \to C^*(G_\Lambda)$ is a homomorphism.

We show that ψ is multiplicative. It suffices to show that $\psi(fg) = \psi(f) * \psi(g)$ for $f \in C_c(\Lambda^m)$ and $g \in C_c(\Lambda^n)$. Indeed, for $(x,p,y) \in G_\Lambda$ we have

$$\begin{aligned}\psi(f) * \psi(g)((x,p,y)) &= \textstyle\sum_{(x,r,w)\in G_\Lambda} \psi(f)((x,r,w))\psi(g)((w,p-r,y)) \\ &= \begin{cases} f(x(0,m))g(x(m,m+n)) & \text{if } p=m+n \text{ and } \sigma^{m+n}(x)=y \\ 0 & \text{otherwise}\end{cases} \\ &= \psi(fg)((x,p,y)).\end{aligned}$$

We now show that $\psi(\langle f,g\rangle_A^n) = \psi(f)^* * \psi(g)$ for $f,g \in X_n$, and for this it is enough to show that $\psi(\langle f,g\rangle_A^n) = \psi(f)^* * \psi(g)$ for $f,g \in C_c(\Lambda^n)$. Noting that $\langle f,g\rangle_A^n \in C_c(\Lambda^0)$ by [21, Lemma 1.5], for $(x,p,y) \in G_\Lambda$ we have

$$\begin{aligned}&\psi(f)^* * \psi(g)((x,p,y)) \\ &\quad = \begin{cases} \sum_{\alpha\in\Lambda^n r(x)} \overline{\psi(f)((\alpha x,n,x))}\psi(g)((\alpha x,n,x)) & \text{if } p=0 \text{ and } x=y \\ 0 & \text{otherwise}\end{cases} \\ &\quad = \begin{cases} \sum_{\alpha\in\Lambda^n r(x)} \overline{f(\alpha)}g(\alpha) & \text{if } p=0 \text{ and } x=y \\ 0 & \text{otherwise}\end{cases} \\ &\quad = \psi(\langle f,g\rangle_A^n)((x,p,y)),\end{aligned}$$

and the result follows.

Uniqueness of the Toeplitz representation ψ follows from the facts that each of the linear maps $\psi|_{X_n}$ is continuous and $C_c(\Lambda^n)$ is dense in X_n.

Finally, to see that $\{\,\psi(f) \mid n \in \mathbb{N}^k, f \in C_c(X_n)\,\}$ generates $C^*(G_\Lambda)$, note that the norm on $C^*(G_\Lambda)$ is dominated by the norm $\|\cdot\|_I$ from [34, Page 50] which in turn is dominated by $\|\cdot\|_\infty$. Our strategy is to use the Stone-Weierstrass Theorem to show that the subalgebra $\mathcal{A}$ generated by $\{\,\psi(f) \mid n \in \mathbb{N}^k, f \in C_c(X_n)\,\}$ is dense in $C_0(G_\Lambda)$. Since $\mathcal{A}$ is by definition a subset of $C_c(G_\Lambda)$, it will then follow that $\mathcal{A}$ is dense in $C_c(G_\Lambda)$, and hence in $C^*(G_\Lambda)$. So it is enough to show that, for distinct $(x,p,y),(x',p',y') \in G_\Lambda$, there exist $m,n \in \mathbb{N}^k$, $f \in C_c(\Lambda^m)$ and $g \in C_c(\Lambda^n)$ such that $\psi(f) * \psi(g)^*((x,p,y)) \neq \psi(f)*\psi(g)^*((x',p',y'))$. So denote $(x,p,y) = (\lambda z, d(\lambda)-d(\mu), \mu z)$. If $p \neq p'$ then choose $f \in C_c(\Lambda^{d(\lambda)})$ with $f(\lambda) = 1$ and $g \in C_c(\Lambda^{d(\mu)})$ with $g(\mu) = 1$. Then

$$\psi(f) * \psi(g)^*((x,p,y)) = \psi(f)((x,d(\lambda),z))\overline{\psi(g)((y,d(\mu),z))} = f(\lambda)\overline{g(\mu)} = 1$$

and $\psi(f) * \psi(g)^*((x',p',y')) = 0$.

Suppose $p = p'$ and $x \neq x'$. Assume without loss of generality that $d(x) \not< d(x')$. If $d(x) = d(x')$ then there exists $m \in \mathbb{N}^k$ such that $d(\lambda) \le m \le d(x)$ and $x(0,m) \neq x'(0,m)$. Choose $f \in C_c(\Lambda^m)$ such that $f(x(0,m)) = 1$ and $f(x'(0,m)) = 0$, and choose $g \in C_c(\Lambda^{m-p})$ such that $g(y(0,m-p)) = 1$. Then, since $\sigma^{m-p}(y) = \sigma^m(x)$,

$$\begin{aligned}\psi(f) * \psi(g)^*((x,p,y)) &= \psi(f)((x,m,\sigma^m(x)))\overline{\psi(g)((y,m-p,\sigma^{m-p}(y)))}\\ &= f(x(0,m))\overline{g(y(0,m-p))} = 1 \quad \text{and}\\ \psi(f) * \psi(g)^*((x',p',y')) &= 0.\end{aligned} \tag{5.6}$$

If $d(x) \neq d(x')$ then there exists $m \in \mathbb{N}^k$ such that $d(\lambda) \le m \le d(x)$ and $m \not\le d(x')$. So, choosing $f \in C_c(\Lambda^m)$ such that $f(x(0,m)) = 1$ and $g \in C_c(\Lambda^{m-p})$ such that $g(y(0,m-p)) = 1$ gives (5.6).

The case where $p = p'$, $x = x'$ and $y \neq y'$ follows from an argument similar to that of the preceding paragraph. □

To show that ψ is Nica covariant we will use the following lemma.

Lemma 5.18. *If $m \in \mathbb{N}^k$, $f,g \in C_c(\Lambda^m)$, and $(x,p,y) \in G_\Lambda$ then*

$$\psi^{(m)}(f \otimes g^*)((x,p,y)) = \begin{cases} f(x(0,m))\overline{g(y(0,m))} & \text{if } p = 0,\ m \le d(x) \text{ and } \sigma^m(x) = \sigma^m(y)\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Recalling that the Haar system on G_Λ consists of counting measures, we calculate

$$\psi^{(m)}(f \otimes g^*)((x,p,y)) = \textstyle\sum_{(x,r,z)\in G_\Lambda} \psi(f)((x,r,z))\psi(g)^*((z,p-r,y)). \tag{5.7}$$

The right-hand side of (5.7) is zero unless $p = 0$, $m \le d(x)$ and $\sigma^m(x) = \sigma^m(y)$ (noting that $p = 0$ implies $d(x) = d(y)$), and if indeed $p = 0$, $m \le d(x)$ and $\sigma^m(x) = \sigma^m(y)$, then (5.7) simplifies to

$$\psi(f)((x,m,\sigma^m(x)))\psi(g)^*((\sigma^m(y),-m,y)) = f(x(0,m))\overline{g(y(0,m))}.$$ □

Proposition 5.19. *Let Λ be a compactly aligned topological k-graph. The Toeplitz representation $\psi\colon X \to C^*(G_\Lambda)$ from Proposition 5.17 is gauge-compatible and Nica covariant.*

Proof. The canonical continuous cocycle $c : (x, m, y) \mapsto m$ on G_Λ induces a coaction β of $\mathbb{Z}^k$ on $C^*(G_\Lambda)$ satisfying $\beta(f) = f \otimes i_{\mathbb{Z}^k}(m)$ whenever $\operatorname{supp}(f) \subset c^{-1}(m)$. In particular, $\beta(\psi(f)) = \psi(f) \otimes i_{\mathbb{Z}^k}(m)$ for $f \in X_m$, so ψ is gauge-compatible.

For Nica covariance, fix $m, n \in \mathbb{N}^k$, $f_m, g_m \in F_m$ and $f_n, g_n \in F_n$. Resume the notation of Lemma 5.14 so that

$$\iota_m^{m\vee n}(g_m \otimes f_m^*)\iota_n^{m\vee n}(f_n \otimes g_n^*) = \textstyle\sum_{i=1}^{r_m} \sum_{j=1}^{r_n} a_{ij} \otimes b_j^*.$$

By Lemma 5.11 it suffices to show that for every $(x, p, y) \in G_\Lambda$,

$$\psi^{(m)}(g_m \otimes f_m^*)\psi^{(n)}(f_n \otimes g_n^*)((x, p, y)) = \psi^{(m\vee n)}(\iota_m^{m\vee n}(g_m \otimes f_m^*)\iota_n^{m\vee n}(f_n \otimes g_n^*))((x, p, y)).$$

By Lemma 5.18, both sides of this equation are equal to zero unless $p = 0$. By definition of the multiplication in $C^*(G_\Lambda)$ and another application of Lemma 5.18, we are left to show that for all $(x, 0, y) \in G_\Lambda$,

$$\begin{aligned}\textstyle\sum_{(x,0,z)\in G_\Lambda} \psi^{(m)}(g_m \otimes f_m^*)((x, 0, z))\psi^{(n)}(f_n \otimes g_n^*)((z, 0, y))& \\ = \psi^{(m\vee n)}(\iota_m^{m\vee n}(g_m \otimes f_m^*)\iota_n^{m\vee n}(f_n \otimes g_n^*))((x, 0, y)).&\end{aligned} \tag{5.8}$$

We may further deduce from Lemma 5.18 that both sides are equal to zero unless $m \vee n \le d(x)$ and $\sigma^{m\vee n}(x) = \sigma^{m\vee n}(y)$ (noting that $p = 0$ implies $d(x) = d(y)$).

So fix $(x, 0, y) \in G_\Lambda$ such that $m \vee n \le d(x)$ and $\sigma^{m\vee n}(x) = \sigma^{m\vee n}(y)$. Recall from Notation 5.10 that for each $h \in F_r$ and $v \in \Lambda^0$ such that $\Lambda^r v$ is non-empty we have a fixed path $\lambda_{h,v} \in \Lambda^r v$ such that $h(\lambda) = 0$ for all $\lambda \in \Lambda^r v \setminus \{\lambda_{h,v}\}$. Let $\lambda_m := \lambda_{f_m,x(m)}$. We calculate, using Lemma 5.18 yet again, that

$$\begin{aligned}&\textstyle\sum_{(x,0,z)\in G_\Lambda} \psi^{(m)}(g_m \otimes f_m^*)((x, 0, z))\psi^{(n)}(f_n \otimes g_n^*)((z, 0, y))\\ &\qquad= \textstyle\sum_{\{\,\zeta\in\Lambda^m x(m) \mid \sigma^n(\zeta\sigma^m(x))=\sigma^n(y)\,\}} g_m(x(0, m))\overline{f_m(\zeta)}f_n([\zeta\sigma^m(x)](0, n))\overline{g_n(y(0, n))}\\ &\qquad= \begin{cases} g_m(x(0, m))\overline{f_m(\lambda_m)}f_n([\lambda_m\sigma^m(x)](0, n))\overline{g_n(y(0, n))} \\ \qquad\qquad \text{if } y(n, m \vee n) = [\lambda_m\sigma^m(x)](n, m \vee n) \\ 0 \qquad\qquad \text{otherwise.}\end{cases}\end{aligned} \tag{5.9}$$

We will show that the right-hand side of (5.8) is equal to (5.9). For $1 \le i \le r_m$ and $1 \le j \le r_n$, we have

$$\begin{aligned}&\psi^{(m\vee n)}(a_{ij} \otimes b_j^*)((x, 0, y))\\ &\qquad= a_{ij}(x(0, m \vee n))\overline{b_j(y(0, m \vee n))}\\ &\qquad= g_m(x(0, m))[\rho_i^m \cdot \langle f_m\rho_i^m, f_n\rho_j^n\rangle_A^{m\vee n}](x(m, m \vee n))\overline{b_j(y(0, m \vee n))}\\ &\qquad= \begin{cases} g_m(x(0, m))[\rho_i^m(x(m, m \vee n))]^2\overline{f_m(\lambda_m)}f_n([\lambda_m\sigma^m(x)](0, n)) \\ \qquad\qquad \overline{g_n(y(0, n))}[\rho_j^n([\lambda_m\sigma^m(x)](n, m \vee n))]^2 \\ \qquad\qquad \text{if } y(n, m \vee n) = [\lambda_m\sigma^m(x)](n, m \vee n) \\ 0 \qquad\qquad \text{otherwise.}\end{cases}\end{aligned}$$

Hence, if $y(n, m \vee n) \neq [\lambda_m \sigma^m(x)](n, m \vee n)$ then each of the right-hand side of (5.8) and (5.9) is equal to zero; and if $y(n, m \vee n) = [\lambda_m \sigma^m(x)](n, m \vee n)$ we calculate:

$$\begin{aligned}
&\psi^{(m \vee n)}(\iota_m^{m \vee n}(g_m \otimes f_m^*)\iota_n^{m \vee n}(f_n \otimes g_n^*))((x, 0, y)) \\
&\qquad = \textstyle\sum_{i=1}^{r_m} \sum_{j=1}^{r_n} \psi^{(m \vee n)}(a_{ij} \otimes b_j^*)((x, 0, y)) \\
&\qquad = \textstyle\sum_{i=1}^{r_m} \sum_{j=1}^{r_n} g_m(x(0, m))[\rho_i^m(x(m, m \vee n))]^2 \overline{f_m(\lambda_m)} \\
&\qquad\qquad\qquad f_n([\lambda_m \sigma^m(x)](0, n)) \overline{g_n(y(0, n))} [\rho_j^n([\lambda_m \sigma^m(x)](n, m \vee n))]^2 \\
&\qquad = g_m(x(0, m)) \overline{f_m(\lambda_m)} f_n([\lambda_m \sigma^m(x)](0, n)) \overline{g_n(y(0, n))}
\end{aligned}$$

as required. □

Theorem 5.20. *Let Λ be a compactly aligned topological k-graph. Let G_Λ and $\mathcal{G}_\Lambda$ be Yeend's path groupoid and boundary-path groupoid for Λ, let $C^*(G_\Lambda)$ and $C^*(\mathcal{G}_\Lambda)$ and $C_r^*(G_\Lambda)$ and $C_r^*(\mathcal{G}_\Lambda)$ be the associated full and reduced C^*-algebras, and let $q\colon C^*(G_\Lambda) \to C^*(\mathcal{G}_\Lambda)$ be the quotient map. Let X be the product system defined by Proposition 5.9 and let $\psi_*\colon \mathcal{T}_{\mathrm{cov}}(X) \to C^*(G_\Lambda)$ be the homomorphism obtained from the universal property of $\mathcal{T}_{\mathrm{cov}}(X)$ and Proposition 5.19.*

Then ψ_* is an isomorphism, the canonical maps $C^*(G_\Lambda) \to C_r^*(G_\Lambda)$ and $C^*(\mathcal{G}_\Lambda) \to C_r^*(\mathcal{G}_\Lambda)$ are isomorphisms, and there is a unique $*$-isomorphism $\phi\colon C^*(\mathcal{G}_\Lambda) \to \mathcal{NO}_X$ which makes the following diagram commute.

$$\begin{array}{ccc}
\mathcal{T}_{\mathrm{cov}}(X) & \xrightarrow{\psi_*} & C^*(G_\Lambda) \\
{\scriptstyle q_{\mathrm{CNP}}}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
\mathcal{NO}_X & \xleftarrow{\phi} & C^*(\mathcal{G}_\Lambda).
\end{array}$$

Proof. Since $\psi(X)$ generates $C^*(G_\Lambda)$, it follows that ψ_* is surjective. Let π_{G_Λ} denote the canonical factor map from $C^*(G_\Lambda)$ to $C_r^*(G_\Lambda)$ and let $\tilde{\psi}_* := \pi_{G_\Lambda} \circ \psi_*$. We aim to show that $\tilde{\psi}_*$ it injective. It will then follow that both ψ_* and π_{G_Λ} are isomorphisms. The canonical continuous cocycle $c : (x, m, y) \mapsto m$ from G_Λ to $\mathbb{Z}^k$ induces a strongly continuous action of $\mathbb{T}^k$ on $C_r^*(G_\Lambda)$ (cf. [34, Proposition II.5.1]) and thereby a coaction β of $\mathbb{Z}^k$ on $C_r^*(G_\Lambda)$ satisfying $\beta(f) = f \otimes i_{\mathbb{Z}^k}(m)$ whenever $\operatorname{supp}(f) \subset c^{-1}(m)$. Thus $\tilde{\psi}_*$ is equivariant for δ and β. Since $\mathbb{Z}^k$ is amenable, the coaction δ is normal, so it is enough to prove that $\ker \tilde{\psi}_* \cap \mathcal{F} = \{0\}$ which we will do now. It follows from equation (3.6), Lemma 3.6 and [1, Lemma 1.3] that it is enough to prove that $\ker \tilde{\psi}_* \cap B_F = \{0\}$ for each $F \in \mathcal{P}_{\mathrm{fin}}^{\vee}(P)$. For this, we fix $F \in \mathcal{P}_{\mathrm{fin}}^{\vee}(P)$ and generalised compact operators $T_p \in \mathcal{K}(X_p)$ for $p \in F$ such that

$$\sum_{p \in F} \tilde{\psi}_*(i_X^{(p)}(T_p)) = 0. \tag{5.10}$$

We use induction over the number of elements in F to show that $T_p = 0$ for each $p \in F$. Proposition 5.17 implies that the representation $\pi_{G_\Lambda} \circ \psi$ is injective. Thus it follows from Lemma 2.4 of [21] that $\tilde{\psi}_* \circ i_X^{(p)} = (\pi_{G_\Lambda} \circ \psi)^{(p)}$ is injective for all $p \in \mathbb{N}^k$. So if $F = \{p\}$, then $T_p = 0$. Assume then that F consists of more than one element. Let p_0 be a minimal element of F. Then $p \not\leq p_0$ for all $p \in F \setminus \{p_0\}$. It follows from Lemma

5.18 that if $(x, 0, y) \in G_\Lambda$ with $d(x) = d(y) = p_0$, then $\tilde{\psi}_*(i_X^{(p)}(T_p))((x, 0, y)) = 0$ for all $p \in F \setminus \{p_0\}$. The assumption (5.10) then implies that $\tilde{\psi}_*(i_X^{(p_0)}(T_{p_0}))((x, 0, y)) = 0$ for all $(x, 0, y) \in G_\Lambda$ with $d(x) = d(y) = p_0$. It then follows from Lemma 5.18 that $\tilde{\psi}_*(i_X^{(p_0)}(T_{p_0}))((x, p, y)) = 0$ for all $(x, p, y) \in G_\Lambda$, and thus that $\tilde{\psi}_*(i_X^{(p_0)}(T_{p_0})) = 0$. As before, this implies that $T_{p_0} = 0$, and it then follows from our inductive hypothesis that $T_p = 0$ for every $p \in F$. Thus $\tilde{\psi}_*$ is injective.

Let $\pi_{\mathcal{G}_\Lambda}$ denote the canonical map from $C^*(\mathcal{G}_\Lambda)$ to $C_r^*(\mathcal{G}_\Lambda)$ and let $\rho : X \to C^*(\mathcal{G}_\Lambda)$ be the Toeplitz representation $\pi_{\mathcal{G}_\Lambda} \circ q \circ \psi$. Then Proposition 5.19 implies that ρ is a Nica covariant representation of X which generates $C_r^*(\mathcal{G}_\Lambda)$. Proposition 4.3 of [37] implies that $\rho_e : A \to C_r^*(\mathcal{G}_\Lambda)$ is injective, so ρ is injective as a representation of X. As above the canonical coaction $(x, m, y) \mapsto m$ from $\mathcal{G}_\Lambda$ to $\mathbb{N}^k$ induces a coaction γ of $\mathbb{Z}^k$ on $C_r^*(\mathcal{G}_\Lambda)$ such that $\gamma(\rho(x)) = \rho(x) \otimes i_{\mathbb{Z}^k}(d(x))$ for $x \in X$. Thus ρ is gauge-compatible. As noted in the proof of Corollary 4.14, a product system over $\mathbb{N}^k$ is automatically $\tilde{\phi}$-injective. Thus it follows from Theorem 4.1 that there exists a surjective $*$-homomorphism $\phi' : C_r^*(\mathcal{G}_\Lambda) \to \mathcal{NO}_X$ such that $\phi' \circ \pi_{\mathcal{G}_\Lambda} \circ q \circ \psi_* = q_{\mathrm{CNP}}$. Let $\phi := \phi' \circ \pi_{\mathcal{G}_\Lambda}$. We will show that ϕ is injective. It will then follow that $\pi_{\mathcal{G}_\Lambda}$ is an isomorphism from $C^*(\mathcal{G}_\Lambda)$ to $C_r^*(\mathcal{G}_\Lambda)$, and that ϕ is an isomorphism from $C^*(\mathcal{G}_\Lambda)$ to $\mathcal{NO}_X$ such that $\phi \circ q \circ \psi_* = q_{\mathrm{CNP}}$. The various maps defined so far are summarised in the following commuting diagram. We have established already that all three maps in the top row are isomorphisms:

$$\begin{tikzcd}
\mathcal{T}_{\mathrm{cov}}(X) \arrow[r, "\psi_*"'] \arrow[rr, bend left, "\tilde{\psi}_*"] \arrow[d, "q_{\mathrm{CNP}}"'] & C^*(G_\Lambda) \arrow[r, "\pi_{G_\Lambda}"'] \arrow[d, "q"'] & C_r^*(G_\Lambda) \\
\mathcal{NO}_X & C^*(\mathcal{G}_\Lambda) \arrow[l, "\phi"'] \arrow[r, "\pi_{\mathcal{G}_\Lambda}"] & C_r^*(\mathcal{G}_\Lambda) \arrow[ll, bend left, "\phi'"]
\end{tikzcd}$$

To show that ϕ is injective it suffices to prove that $\ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \subset \ker(q \circ \pi_{G_\Lambda}^{-1})$. Since $\mathcal{G}_\Lambda = G_{\Lambda|\partial\Lambda}$ it follows that $\ker(q \circ \pi_{G_\Lambda}^{-1})$ is the closure of

$$\{f \in C_c(G_\Lambda) \mid f((x, m, y)) = 0 \text{ if } (x, m, y) \in G_\Lambda \text{ and } x, y \in \partial\Lambda\}.$$

It follows that there is an approximate identity for $\ker(q \circ \pi_{G_\Lambda}^{-1})$ in $C_0(G_\Lambda^{(0)} \setminus \partial\Lambda)$, and $\ker(q \circ \pi_{G_\Lambda}^{-1})$ is therefore generated by its intersection with $C_0(G_\Lambda^{(0)})$, and thus by its intersection with $C_r^*(G_\Lambda[c])$ where $G_\Lambda[c]$ is the subgroupoid $c^{-1}(\{0\}) = \{(x, m, y) \in G_\Lambda \mid m = 0\}$. Thus it follows from Proposition A.1 that β induces a coaction $\beta^{\ker(q \circ \pi_{G_\Lambda}^{-1})}$ of $\mathbb{Z}^k$ on $C^*(\mathcal{G}_\Lambda)$, which is normal since $\mathbb{Z}^k$ is amenable. Since $q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}$ is equivariant for β and ν, it follows that it suffices to show that $\ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \cap C_r^*(G_\Lambda[c]) \subset \ker(q \circ \pi_{G_\Lambda}^{-1})$.

By [37, Theorem 3.16], G_Λ, and thus $G_\Lambda[c]$, are r-discrete, and since $G_\Lambda[c]$ is also (essentially) principal, [34, Proposition II.4.6] implies that

$$Y := \{x \in G_\Lambda^{(0)} \mid f((x, 0, x)) = 0 \text{ for all } f \in \ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \cap C_r^*(G_\Lambda[c])\}$$

is a closed $G_\Lambda[c]$-invariant subset of $G_\Lambda^{(0)}$ such that $\ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \cap C_r^*(G_\Lambda[c])$ is the closure of

$$\{f \in C_c(G_\Lambda[c]) \mid f((x,0,y)) = 0 \text{ if } (x,0,y) \in G_\Lambda[c] \text{ and } x,y \in Y\}.$$

We claim that Y is not just $G_\Lambda[c]$-invariant, but also G_Λ-invariant; indeed if $(x,m,y) \in G_\Lambda$ and $x \notin Y$, then there exists $f \in \ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \cap C_r^*(G_\Lambda[c])$ such that $f((x,0,x)) \neq 0$. Since G_Λ is r-discrete there is $g \in C_c(G_\Lambda)$ such that g is supported on a subset on which the source map is bijective, and such that $g((x,m,y)) = 1$. We then have that

$$\begin{aligned}
&(g^* * f * g)((y,0,y)) \\
&\quad = \textstyle\sum_{(y,m_1,z_1),(z_2,m_2,y) \in G_\Lambda} \overline{g((z_1,-m_1,y))} f((z_1,-m_1-m_2,z_2)) g((z_2,m_2,y)) \\
&\quad = \overline{g((x,m,y))} f((x,0,x)) g((x,m,y)) = f((x,0,x)) \neq 0.
\end{aligned}$$

Let Φ^β be the conditional expectation of $C_r^*(G_\Lambda)$ onto $C_r^*(G_\Lambda[c])$ induced by the coaction β. Then $\Phi^\beta(g^* * f * g)((y,0,y)) = (g^* * f * g)((y,0,y)) \neq 0$, and since $\ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1})$ is generated by its intersection with the subset $\tilde{\psi}_*(\mathcal{F})$ of $C_r^*(G_\Lambda[c])$ it follows from [14, Theorem 3.9] that $\Phi^\beta(g^* * f * g) \in \ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \cap C_r^*(G_\Lambda[c])$. Thus $y \notin Y$, showing that Y is G_Λ-invariant. Since $q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}$ is injective on $\pi_{G_\Lambda}(\psi(C_0(\Lambda^{(0)})))$ we must have that $vY \neq \emptyset$ for all $v \in \Lambda^0$. Thus Y is a closed and invariant subset of $G_\Lambda^{(0)}$ which satisfies that $vY \neq \emptyset$ for all $v \in \Lambda^0$. It therefore follows from Proposition 5.16 that $\partial\Lambda \subset Y$. Thus $\ker(q_{\mathrm{CNP}} \circ \tilde{\psi}_*^{-1}) \cap C_r^*(G_\Lambda[c]) \subset \ker(q \circ \pi_{G_\Lambda}^{-1})$, as claimed. $\square$

By combining Theorem 5.20 with Corollary 4.14 we get the following gauge-invariant uniqueness result for $C^*(\mathcal{G}_\Lambda)$.

Corollary 5.21. *Let Λ be a compactly aligned topological k-graph. Let $\mathcal{G}_\Lambda$ be Yeend's boundary-path groupoid for Λ, and let $C^*(\mathcal{G}_\Lambda)$ be the associated full C^*-algebra. Let $\psi\colon X \to C^*(G_\Lambda)$ be the map from Proposition 5.17.*

A surjective $*$-homomorphism $\phi : C^*(\mathcal{G}_\Lambda) \to B$ is injective if and only if

1. *there is a strongly continuous action α of $\mathbb{T}^k$ on B such that $\alpha_z(\phi(\psi(x)) = z^{d(x)}\phi(\psi(x))$ for all $x \in X$ and $z \in \mathbb{T}^k$, and*
2. *$\phi|_{\psi(C_0(\Lambda^0))}\colon A \to B$ is injective.*

We note that Yamashita in [36] has studied $\mathcal{NO}_X$ under the assumption that Λ is row-finite and without sources. Among other things he shows a Cuntz-Krieger type uniqueness theorem for $\mathcal{NO}_X$ and gives sufficient conditions for when $\mathcal{NO}_X$ is simple and purely infinite.

Appendix A. Coactions, quotients and normality

Proposition A.1. *Let A be a C^*-algebra carrying a coaction δ of a discrete group G. Let I be an ideal of A which is generated as an ideal by $I_e^\delta := I \cap A_e^\delta$. Let $q_I\colon A \to A/I$ be the quotient map. Then there is a coaction δ^I of G on A/I such that*

$$\delta^I \circ q_I = (q_I \otimes \mathrm{id}_{C^*(G)}) \circ \delta. \tag{A.1}$$

Proof. The proposition is trivially true if $A = \{0\}$, so we may, and will, assume that A contains a non-zero element.

To define the homomorphism δ^I, observe that for $a \in A_e^\delta$, we have $(q_I \otimes \mathrm{id}_{C^*(G)}) \circ \delta(a) = q_I(a) \otimes i_G(e)$, which is equal to zero if and only if $a \in I$; that is $\ker((q_I \otimes \mathrm{id}_{C^*(G)}) \circ \delta) \cap A_e^\delta = I_e^\delta$. Since I is generated as an ideal by I_e^δ, it follows that $I \subset \ker(q_I \otimes \mathrm{id}_{C^*(G)}) \circ \delta)$, and hence $(q_I \otimes \mathrm{id}_{C^*(G)}) \circ \delta$ descends to a homomorphism $\delta^I \colon A/I \to (A/I) \otimes C^*(G)$ satisfying (A.1).

We will show that δ^I is a coaction. Since G is discrete, it suffices to show that δ^I is nondegenerate and injective and satisfies the coaction identity (see [29, Section 1]). It follows from [29, Corollary 1.6] that A_e^δ contains an approximate identity $(u_\lambda)_{\lambda \in \Lambda}$ for A. To see that δ^I is nondegenerate, note that $(q_I(u_\lambda))_{\lambda \in \Lambda}$ is an approximate identity for A/I. Since each $u_\lambda \in A_e^\delta$, we have $\delta^I(q_I(u_\lambda)) = q_I(u_\lambda) \otimes \mathrm{id}_{C^*(G)}$ for all λ, and hence $(\delta^I(u_\lambda))_{\lambda \in \Lambda}$ is an approximate identity for $(A/I) \otimes C^*(G)$. So δ^I is nondegenerate.

To see that δ is injective, let $\epsilon \colon C^*(G) \to \mathbb{C}$ be the augmentation representation $i_G(g) \mapsto 1$ for all $g \in G$. For $g \in G$ and $a \in A_g^\delta$, we have

$$(\mathrm{id}_{A/I} \otimes \epsilon) \circ \delta^I(q_I(a)) = (\mathrm{id}_{A/I} \otimes \epsilon)(q_I(a) \otimes i_G(g)) = q_I(a) \otimes 1.$$

Lemma 1.5 of [29] shows that $A = \overline{\mathrm{span}}\,(\bigcup_{g \in G} A_g^\delta)$, so the preceding calculation together with linearity and continuity of the homomorphism $(\mathrm{id}_{A/I} \otimes \epsilon) \circ \delta$ show that

$$(\mathrm{id}_{A/I} \otimes \epsilon) \circ \delta^I(x) = x \otimes 1$$

for all $x \in A/I$. Hence $(\mathrm{id}_{A/I} \otimes \epsilon) \circ \delta^I$ is injective, and in particular, δ^I is injective.

To see that δ^I satisfies the coaction identity, let δ_G be the comultiplication on $C^*(G)$, fix $g \in G$ and $a \in A_g^\delta$, and calculate

$$(\mathrm{id}_{A/I} \otimes \delta_G) \circ \delta^I(q_I(a)) = q_I(a) \otimes i_G(g) \otimes i_G(g) = (\delta^I \otimes \mathrm{id}_{C^*(G)}) \circ \delta^I(q_I(a)).$$

It then follows from linearity and continuity that $(\mathrm{id}_{A/I} \otimes \delta_G) \circ \delta^I = (\delta^I \otimes \mathrm{id}_{C^*(G)}) \circ \delta^I$. We have now established that δ^I is a coaction. □

Remark A.2. One could also prove Proposition A.1 by using the duality between coactions of G and topological G-gradings (cf. [29]) and [14, Proposition 3.11].

Notation A.3. Given a discrete group G, we will write λ_G for the left regular representation of G on $\ell^2(G)$, and also for the resulting homomorphism of $C^*(G)$ onto $C_r^*(G)$ obtained from the universal property of $C^*(G)$ applied to λ_G.

Lemma A.4. *Resume the hypotheses of Proposition A.1. The following are equivalent.*

(1) δ^I *is normal.*
(2) Φ^{δ^I} *is faithful on positive elements.*
(3) $(\mathrm{id}_{A/I} \otimes \lambda_G) \circ \delta^I$ *is injective.*
(4) $\ker((q_I \otimes \lambda_G) \circ \delta) = I$.

Proof. The equivalence of (1) and (2) is an application of [29, Lemma 1.4]. The equivalence of (1) and (3) is by definition of normality; see [11, Definitions A.39 and A.50]. To establish the equivalence of (4) and (3), just observe that

$$(q_I \otimes \lambda_G) \circ \delta = (\mathrm{id}_{A/I} \otimes \lambda_G) \circ (q_I \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_{A/I} \otimes \lambda_G) \circ \delta^I \circ q_I. \qquad \square$$

Recall that a discrete group G is called *exact* if its reduced C^*-algebra $C_r^*(G)$ is exact.

Proposition A.5. *Let G be a discrete group. Then the following are equivalent:*

(1) *G is exact.*
(2) *For every normal coaction δ of G on a C^*-algebra A, and every ideal I of A which is generated by its intersection with A_e^δ, the induced coaction δ^I of G on A/I is normal.*

Proof. Assume that G is not exact. Then there exists a C^*-algebra A which has an ideal I such that $I \otimes C_r^*(G) \subsetneq \ker(q_I \otimes \mathrm{id}_{C_r^*(G)})$ where $q_I\colon A \to A/I$ is the quotient map. Let δ_G denote the coaction of G on $C_r^*(G)$ given by $\delta_G(\lambda_G(g)) = \lambda_G(g) \otimes i_G(g)$ for all $g \in G$ (see [30, Example 1.15] or [28, Proposition 2.4]). The $*$-homomorphism $\delta := \mathrm{id}_A \otimes \delta_G$ is then a coaction of G on $A \otimes C_r^*(G)$. Let $\Delta\colon C_r^*(G) \to C_r^*(G) \otimes C_r^*(G)$ be given by $\Delta(x) = x \otimes x$ for all $x \in C_r^*(G)$. We then have that $(\mathrm{id}_{A\otimes C_r^*(G)} \otimes \lambda_G) \circ \delta = \mathrm{id}_A \otimes \Delta$, from which it follows that $(\mathrm{id}_{A\otimes C_r^*(G)} \otimes \lambda_G) \circ \delta$ is injective, and thus that δ is normal. It is easy to check that $(A \otimes C_r^*(G))_e^\delta = A \otimes 1_{C_r^*(G)}$, and that the ideal $I \otimes C_r^*(G)$ of $A \otimes C_r^*(G)$ is generated by its intersection with $A \otimes \mathrm{id}_{C_r^*(G)}$. We have that $(q_{I\otimes C_r^*(G)} \otimes \lambda_G) \circ \delta = q_I \otimes \Delta$ from which it follows that

$$I \otimes C_r^*(G) \subsetneq \ker(q_I \otimes \mathrm{id}_{C_r^*(G)}) \subset \ker(q_I \otimes \Delta) = \ker((q_{I\otimes C_r^*(G)} \otimes \lambda_G) \circ \delta),$$

so it follows from Lemma A.4 that $\delta^{I\otimes C_r^*(G)}$ is not normal.

Assume now that G is exact and let δ be a coaction of G on a C^*-algebra A, and I an ideal of A which is generated by its intersection with A_e^δ. If $x \in \ker((q_I \otimes \lambda_G) \circ \delta)$, then $(\mathrm{id}_A \otimes \lambda_G)(\delta(x)) \ker(q_I \otimes \mathrm{id}_{C_r^*(G)}) = I \otimes C_r^*(G)$, from which it follows that $x \in I$. Thus $\ker((q_I \otimes \lambda_G) \circ \delta) = I$, so it follows from Lemma A.4 that δ^I is normal. □

Remark A.6. The first half of the proof is essentially taken from [14, page 61], and is adapted to our coaction framework.

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